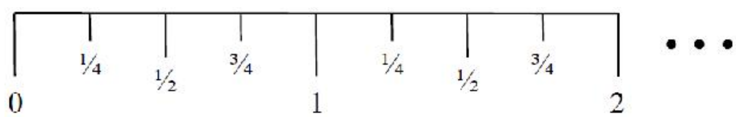
University of Northern Colorado Mathematics Contest 2016-2017

Problems of First Round

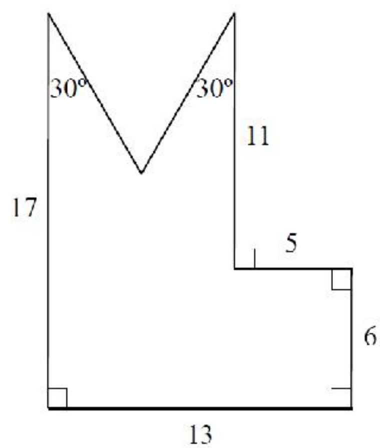
1. A snail crawls on a vertical rock face that is 5 feet high. The snail climbs up 3 feet in a day and then rests through the night. Each night it slides down 2 feet while it rests. If it starts at the bottom on the morning of September 1, on what day of the month does it first reach the top of the rock face?

2. (a) A tape measure's short, medium, and long marks indicate quarter-inches, half-inches, and whole inches, respectively. How many short marks fall between the 1-inch mark and the 7-inch mark?

   (b) How many short marks fall between the 11-inch mark and the 413-inch mark?



3. What is the perimeter (that is, the sum of the lengths of the sides) of the figure? The lengths of the lower sides are 17, 13, 6, 5, and 11. The two acute angles at the top each measure 30 degrees. There are four right angled corners in the figure, indicated by the small squares.



4. A hog trading team sells two hogs for $120 each. They sell one of the hogs for 125% of the price they paid for the hog. They sell the other hog for 80% of the price they paid for it. Do they make a profit or a loss overall, and how much, in dollars, is that profit or loss?

5. A box of 500 balls contains balls numbered $1, 2, 3, \cdots, 100$ in each of five different colors. Without ever looking at any of the balls, you are to choose balls at random from the box and put them into a bag. If you must be sure that when you finish, the bag contains at least one set of five balls with identical numbers, then what is the smallest number of balls that you can put in the bag?

6. Tutu, Jada, and Faith eat lunch together. Tutu contributes 9 sausages, Jada contributes 8 sausages, and the three girls divide the sausages equally. Faith has brought no food, but gives the other two girls 17 wupiupi coins in exchange for her share of the sausages. How many of the coins should Tutu get?

7. What is the first time after midnight at which the hour hand and minute hand on an ordinary clock face are perpendicular to one another? Express the time in the format Hour, Minute, Second, with your answer rounded to the nearest second. Assume the clock is a 12 hour clock with hands that move at uniform speeds.

8. How many integers greater than 0 and less than 100, 000 are palindromes? An integer is a palindrome if its digits are the same when read left to right and right to left. For instance, 2134312 and 353 are palindromes; so are 1001, 99, 5, and 1. Reminder: do not count the number 0.

9. A polynomial $P(x)$ satisfies the equation

$$P(P(x)-1)=1+x^{16}.$$

What is $P(2)$? (The expression $P(P(x)-1)$ on the left side of the equation means "plug $P(x)-1$ into $P(x)$." The parentheses in this case do not indicate multiplication.)

10. How many different case-sensitive passwords can be created with at most 8 keystrokes, if each keystroke may touch either the "caps lock" key or any of the 10 alphabetic keys on the top row of the keyboard: Q W E R T Y U I O P? Assume that password entry always begins with "caps lock" in lowercase mode, and assume that a password must contain at least one letter. Tapping the "caps lock" key toggles the mode of the keyboard between lowercase and uppercase. Assume that holding a key down does not produce multiple copies of a letter; that is, in the password field, holding a key down has no effect.

University of Northern Colorado Mathematics Contest 2016-2017

Problems and Solutions of First Round

1. A snail crawls on a vertical rock face that is 5 feet high. The snail climbs up 3 feet in a day and then rests through the night. Each night it slides down 2 feet while it rests. If it starts at the bottom on the morning of September 1, on what day of the month does it first reach the top of the rock face?

Answer: September 3

Solution:

At the end of the day of September 1, it will be at 3 feet away from the bottom.

In the morning of September 2, it will be at 1 foot away from the bottom.

At the end of the day of September 2, it will be at 4 feet away from the bottom.

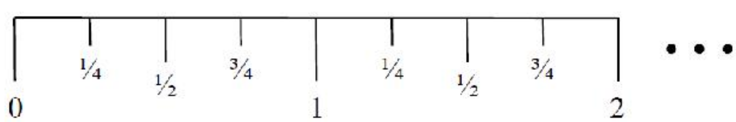
In the morning of September 3, it will be at 2 feet away from the bottom.

At the end of the day of September 3, it will be at 5 feet away from the bottom. The snail is at the top.

The answer is September 3.

2. (a) A tape measure's short, medium, and long marks indicate quarter-inches, half-inches, and whole inches, respectively. How many short marks fall between the 1-inch mark and the 7-inch mark?

   (b) How many short marks fall between the 11-inch mark and the 413-inch mark?



Answer: (a) 12; (b) 804

Solution:

In one whole inch there are two short marks.

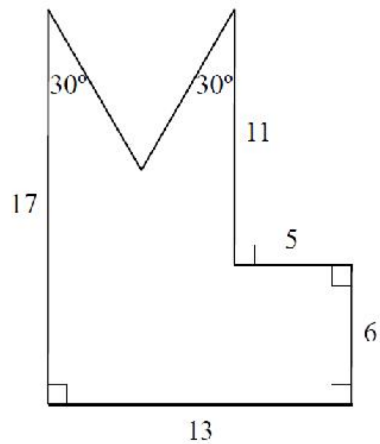
(a) From the 1-in mark to the 7-inch mark there are $7-1=6$ whole inches.

The answer is $6 \cdot 2 = 12$.

(b) From the 11-in mark to the 413-inch mark there are $413-11=402$ whole inches.

The answer is $402 \cdot 2 = 804$.

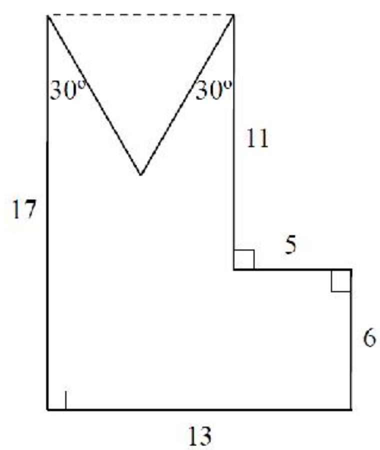
3. What is the perimeter (that is, the sum of the lengths of the sides) of the figure? The lengths of the lower sides are 17, 13, 6, 5, and 11. The two acute angles at the top each measure 30 degrees. There are four right angled corners in the figure, indicated by the small squares.



Answer: 68

Solution:

Connect the two top points.



Since $17 = 11 + 6$, the top dotted line is parallel to the bottom line. The length of the top dotted segment is $13 - 5 = 8$.

The triangle is equilateral with the given two angles of 30°.

Therefore, the perimeter of the original polygon is

$$17 + 8 + 8 + 11 + 5 + 6 + 13 = 68 .$$

4. A hog trading team sells two hogs for \$120 each. They sell one of the hogs for 125% of the price they paid for the hog. They sell the other hog for 80% of the price they paid for it. Do they make a profit or a loss overall, and how much, in dollars, is that profit or loss?

Answer: loss of \$6.

Solution:

The team paid $\frac{120}{1.25}+\frac{120}{0.8}=96+150=246$ dollars for the two hogs.

They made $120\cdot 2=240$ dollars by selling the two hogs.

They lost \$6.

5. A box of 500 balls contains balls numbered 1, 2, 3, $\cdots$, 100 in each of five different colors. Without ever looking at any of the balls, you are to choose balls at random from the box and put them into a bag. If you must be sure that when you finish, the bag contains at least one set of five balls with identical numbers, then what is the smallest number of balls that you can put in the bag?

Answer: 401

Solution:

In the worst case we don't have 5 balls with identical numbers if we already have $100\cdot 4=400$ balls with 4 balls of each number. However, we will have 5 balls of some number if we take one more ball.

The answer is $100\cdot 4+1=401$.

Practice Problem:

A box of 500 balls contains balls numbered 1, 2, 3, $\cdots$, 100 in each of five different colors. Without ever looking at any of the balls, you are to choose balls at random from the box and put them into a bag. If you must be sure that when you finish, the bag contains at least one set of five balls of the same color, then what is the smallest number of balls that you can put in the bag?

Answer: 21

Solution:

In the worst case we don't have 5 balls of the same color if we already have $5\cdot 4=20$ balls with 4 balls of each color. However, we will have 5 balls of some color if we take one more ball.

The answer is $5\cdot 4+1=21$.

6. Tutu, Jada, and Faith eat lunch together. Tutu contributes 9 sausages, Jada contributes 8 sausages, and the three girls divide the sausages equally. Faith has brought no food, but gives the other two girls 17 wupiupi coins in exchange for her share of the sausages. How many of the coins should Tutu get?

Answer: 10 wupiupi coins

Solution:

The total number of sausages is $9+8=17$. Each gets $\frac{17}{3}$ sausages. So Tutu gives $9-\frac{17}{3}=\frac{10}{3}$ to Faith, and Jada gives $8-\frac{17}{3}=\frac{7}{3}$ to Faith. They should distribute Faith's 17 wupiupi coins in this ratio: $\frac{10}{3}:\frac{7}{3}=10:7$.

Therefore, Tutu gets 10 wupiupi coins.

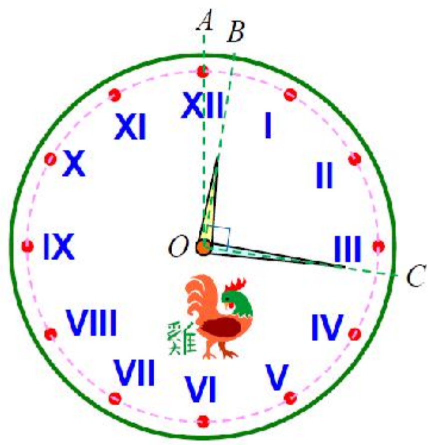
7. What is the first time after midnight at which the hour hand and minute hand on an ordinary clock face are perpendicular to one another? Express the time in the format Hour, Minute, Second, with your answer rounded to the nearest second. Assume the clock is a 12 hour clock with hands that move at uniform speeds.

Answer: 12:16:22

Solution:

Let O be the center of the clock. OA is the ray pointing to 12 o'clock. OB is the ray on which the hour hand lies, and OC is the ray on which the minute hand lies.

Let the desired time be x minutes past 12 o'clock.



Note that x minutes = $\frac{x}{60}$ hours.

From 12 o'clock the minute hand moves for x minutes. The minute hand moves 6° per minute. So $\angle AOC = 6x$ in degrees.

From 12 o'clock the hour hand moves for $\frac{x}{60}$ hours. The hour hand moves 30° per hour. So $\angle AOB = \frac{x}{60}\cdot 30 = \frac{x}{2}$ in degrees.

We have the following equation

$$\frac{x}{2}+90=6x .$$

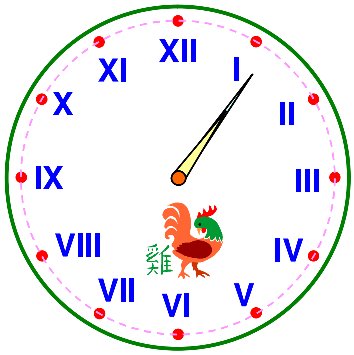
Solving for x we obtain $x=\frac{180}{11}$.

$\frac{180}{11}$ minutes ≈ 16 minutes and 22 seconds rounding to the nearest second.

The desired time is 12:16:22.

Practice Problem 1:

Between 1 o'clock and 2 o'clock the hour hand and minute hand coincide as shown. What is the exact time?

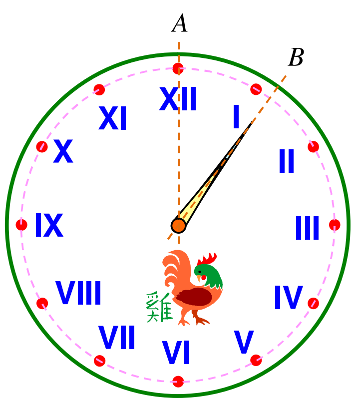


Answer: $5\frac{5}{11}$ minutes past one

Solution:

Let the two hands coincide at x minutes past one.

Note that x minutes = $\frac{x}{60}$ hours.



From 12 o'clock the minute hand moves for x minutes. The minute hand moves 6° per minute. So $\angle AOB = 6x$ in degrees.

From 12 o'clock the hour hand moves for $1 + \frac{x}{60}$ hours. The hour hand moves 30° per hour. So $\angle AOB = \left(1 + \frac{x}{60}\right) \cdot 30 = 30 + \frac{x}{2}$ in degrees.

We have the equation

$$30 + \frac{x}{2} = 6x\,.$$

Solving for x we obtain

$$x = \frac{60}{11} = 5\frac{5}{11}\,.$$

The hour hand and minute hand coincide at $5\frac{5}{11}$ minutes past one.

Practice Problem 2:

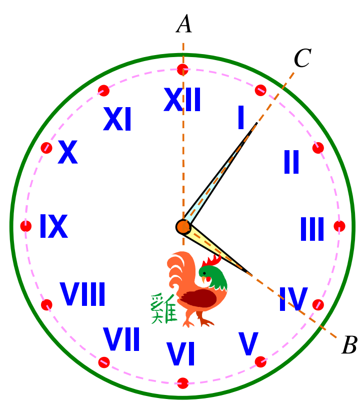
Between 4 o'clock and 5 o'clock there are two moments such that the hour hand and minute hand intersect at a right angle. Find the exact times for the two moments.

Answer: $5\frac{5}{11}$ minutes past four and $38\frac{2}{11}$ minutes past four

Solution:

Let the time be x minutes past four.

After several minutes past four there is a moment when the hour hand and minute hand intersect at a right angle:



Look at the hour hand: $\angle AOB = \left(4 + \frac{x}{60}\right) \cdot 30 = 120 + \frac{x}{2}$ in degrees.

Look at the minute hand: $\angle AOC = 6x$ in degrees.

With $\angle AOB = 90 + \angle AOC$ in degrees we have

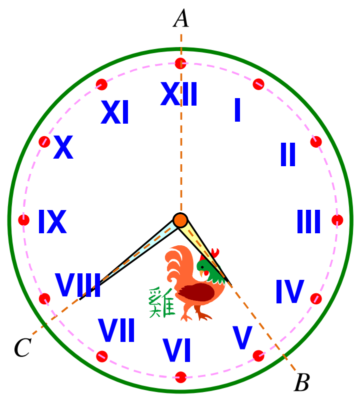
$$120 + \frac{x}{2} = 6x + 90\,.$$

Solving for x we obtain

$$x = \frac{60}{11} = 5\frac{5}{11}\,.$$

The time is $5\frac{5}{11}$ minutes past four.

After some minutes past four and a half there is a moment when the hour hand and minute hand intersect at a right angle:



Look at the hour hand: $\angle AOB = \left(4 + \frac{x}{60}\right) \cdot 30 = 120 + \frac{x}{2}$ in degrees.

Look at the minute hand: $\angle AOC = 6x$ in degrees.

Now $\angle AOC = \angle AOB + 90$ in degrees we have

$$120 + \frac{x}{2} + 90 = 6x$$

Solving for x we obtain

$$x = \frac{420}{11} = 38\frac{2}{11}.$$

The time is $38\frac{2}{11}$ minutes past four.

8. How many integers greater than 0 and less than 100,000 are palindromes? An integer is a palindrome if its digits are the same when read left to right and right to left. For instance, 2134312 and 353 are palindromes; so are 1001, 99, 5, and 1. Reminder: do not count the number 0.

Answer: 1098

Solution:

There are 9 one-digit palindromes: $1, 2, \cdots, 9$.

For two-digit palindromes, the units digit is determined by the tens digit. There are 9 choices ($1, 2, \cdots, 9$) for the tens digit. So there are 9 two-digit palindromes: $11, 22, \cdots, 99$.

For three-digit palindromes, the units digit is determined by the hundreds digit. There are 9 choices ($1, 2, \cdots, 9$) for the hundreds digit, and there are 10 choices ($0, 1, \cdots, 9$) for the tens digit. So there are $9 \cdot 10 = 90$ three-digit palindromes.

For four-digit palindromes, the units digit is determined by the thousands digit, and the tens digit is determined by the hundreds digit. There are 9 choices ($1, 2, \cdots, 9$) for the thousands digit, and there are 10 choices (0, 1, $\cdots$, 9) for the hundreds digit So there are $9 \cdot 10 = 90$ four-digit palindromes.

Similarly, there are $9 \cdot 10 \cdot 10 = 900$ five-digit palindromes.

Therefore, the number of palindromes greater than 0 and less than 100,000 is

$$2 \cdot 9 + 2 \cdot 90 + 900 = 1098 .$$

9. A polynomial $P(x)$ satisfies the equation

$$P(P(x)-1) = 1 + x^{16} .$$

What is $P(2)$? (The expression $P(P(x)-1)$ on the left side of the equation means "plug $P(x)-1$ into $P(x)$." The parentheses in this case do not indicate multiplication.)

Answer: 17

Solution:

Rewrite the given equation:

$$P(P(x)-1)-1 = x^{16} .$$

Let $Q(x) = P(x)-1$, which is also a polynomial. We have

$$Q(Q(x)) = x^{16}$$

Now we can easily see $Q(x) = x^4$ (see the appendix for the proof). Then $P(x) = Q(x)+1 = x^4 + 1$.

Therefore, $P(2) = 2^4 + 1 = 17$.

Appendix:

If $Q(x)$ is a polynomial and $Q(Q(x)) = x^{16}$, then $Q(x) = x^4$. Prove it.

Proof:

Let *n* be the degree of $Q(x)$ where *n* is a nonnegative integer. That is, $Q(x) = ax^n + \cdots$.

Then $Q(Q(x)) = a(ax^n + \cdots)^n + \cdots$.

In the right side the leading term is $a^{n+1}x^{n^2}$. We must have $n^2 = 16$ and $a^{n+1} = 1$.

Since *n* is a non-negative integer, $n = 4$. And then $a = 1$.

Let $Q(x) = x^4 + bx^3 + cx^2 + dx + e$.

We claim that $b = 0$.

Let $Q(x) = x^4 + bx^3 + R(x)$ where $R(x) = cx^2 + dx + e$. Then

$$x^{16} = Q(Q(x)) = (x^4 + bx^3 + R(x))^4 + b(x^4 + bx^3 + R(x))^3 + R(x^4 + bx^3 + R(x)).$$

We have

$$0 = 4bx^{15} + \cdots .$$

So $b = 0$. Then $Q(x) = x^4 + cx^2 + dx + e$.

Similarly, $c = d = e = 0$.

Therefore, $Q(x) = x^4$.

10. How many different case-sensitive passwords can be created with at most 8 keystrokes, if each keystroke may touch either the "caps lock" key or any of the 10 alphabetic keys on the top row of the keyboard: Q W E R T Y U I O P? Assume that password entry always begins with "caps lock" in lowercase mode, and assume that a password must contain at least one letter. Tapping the "caps lock" key toggles the mode of the keyboard between lowercase and uppercase. Assume that holding a key down does not produce multiple copies of a letter; that is, in the password field, holding a key down has no effect.

Answer: 204768420

Solution:

My counting is based on the understanding that the series of "caps lock" "Q", "Q", "Q", "caps lock" with 5 keystrokes produces the same password as the series of "caps lock", "caps lock", "caps lock", "Q", "Q", "Q" with 6 keystrokes as an example.

Case 1: passwords of length *k* ($k = 1, 2, 3, 4$)

Any of the *k* letters in the passwords can be any of the 10 letters in both cases. So there are 20^k passwords of length *k* ($k = 1, 2, 3, 4$).

In total there are $20^1 + 20^2 + 20^3 + 20^4 = 168420$ possible passwords.

Case 2: passwords of length 5

There are 5 letters in the passwords. We may have no press of "caps lock". Before any of the five letters we may insert one "caps lock". We may choose two letters before each of which we insert one "caps lock". We may choose three letters before each of which we insert one "caps lock".

The number of the possible passwords in this case is $10^5\left(1 + \binom{5}{1} + \binom{5}{2} + \binom{5}{3}\right) = 2600000$.

Case 3: passwords of length 6

Similarly, the number of the possible passwords in this case is $10^6\left(1 + \binom{6}{1} + \binom{6}{2}\right) = 22000000$.

Case 4: passwords of length 7

The number of the possible passwords in this case is $10^7\left(1 + \binom{7}{1}\right) = 80000000$.

Case 5: passwords of length 8

The number of the possible passwords in this case is $10^8 = 100000000$.

The total number of passwords which may be created is

$$168420 + 2600000 + 22000000 + 80000000 + 100000000 = 204768420.$$

Practice Problem:

How many different case-sensitive passwords can be created with exactly 8 keystrokes, if each keystroke may touch either the “caps lock” key or any of the 10 alphabetic keys on the top row of the keyboard: Q W E R T Y U I O P? Assume that password entry always begins with "caps lock" in lowercase mode, and assume that a password must contain at least one letter and a password must begin with a capital letter. Tapping the “caps lock” key toggles the mode of the keyboard between lowercase and uppercase. Assume that holding a key down does not produce multiple copies of a letter; that is, in the password field, holding a key down has no effect. The order of “caps lock” matters. For example, the series of 8 presses: “caps lock”, “caps lock”, “caps lock”, “Q”, “Q”, “Q”, “caps lock”, “caps lock” produces a different password as a series of “caps lock”, “Q”, “Q”, “Q”, “caps lock”, “caps lock”, “caps lock”, “caps lock”.

Answer: 17863240

Solution:

The first press (keystroke) must be the “caps lock”.

The second press has 11 choices: any of 10 letters and the “caps lock”.

For any second press with a letter, the following 6 keystrokes can be any of 11 possible presses. So there are $10 \cdot 11^6$ passwords.

If the second press is the “caps lock”, it comes to the lowercase mode. To produce a password, the third press must be the “caps lock”.

Then the fourth press has 11 choices: any of 10 letters and the “caps lock”.

For any fourth press with a letter, the following 4 keystrokes can be any of 11 possible presses.

So there are $10 \cdot 11^4$ passwords.

Continue this pattern. The total number of passwords is

$$10 \cdot 11^6 + 10 \cdot 11^4 + 10 \cdot 11^2 + 10 = 17863240 .$$