Twenty-fifth Annual UNC Math Contest First Round Solutions Fall, 2016

Rules: 90 minutes; no electronic devices.

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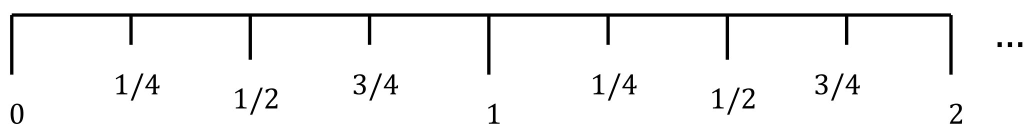
1. A snail crawls on a vertical rock face that is 5 feet high. The snail climbs up 3 feet in a day and then rests through the night. Each night it slides down 2 feet while it rests. If it starts at the bottom on the morning of September 1, on what day of the month does it first reach the top of the rock face?

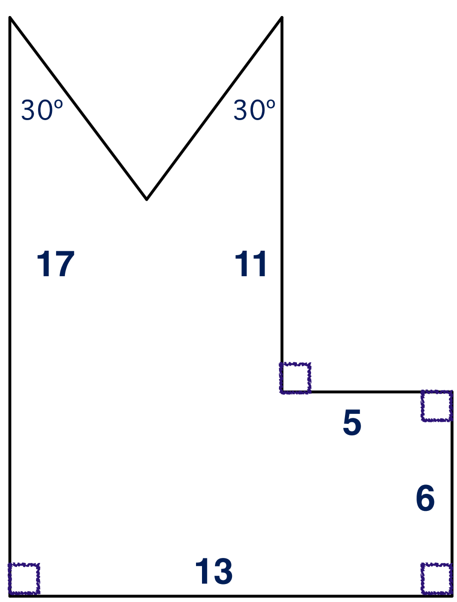
Answer: September 3
Solution: September 1 the snail gets 3 feet high. That night it slides back to 1 foot mark. September 2 the snail gets $1 + 3 = 4$ feet high. That night it slides back to $4 - 3 = 2$ foot mark. September 3 the snail gets $2 + 3 = 5$ feet high.



2. (a) A tape measure's short, medium, and long marks indicate quarter-inches, half-inches, and whole inches, respectively. How many short marks fall between the 1-inch mark and the 7-inch mark?

(b) How many short marks fall between the 11-inch mark and the 413-inch mark?

Answer: (a) 12 (b) 804
Solution: (a) Draw a sketch and count, or reason that there are 6 whole inches between the 1-inch mark and the 7-inch mark and each whole inch contains 2 of the small quarter-inch marks. (b) There are 402 whole inches between the 11-inch mark and the 413-inch mark and therefore there are $2 \times 402 = 804$ small marks.



3. What is the perimeter (that is, the sum of the lengths of the sides) of the figure? The lengths of the lower sides are $17, 13, 6, 5$, and 11. The two acute angles at the top each measure 30 degrees. There are four right angled corners in the figure, indicated by the small squares.

Answer: 68
Solution: The triangle cut out at the top is equilateral with side length $13 - 5 = 8$. The perimeter of the given figure is the sum of the given side lengths plus the lengths of the two bottom sides of the triangle, or $17 + 13 + 6 + 5 + 11 + 8 + 8 = 68$.

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4. A hog trading team <u>sells</u> two hogs for \$120 each. They sell one of the hogs for 125% of the price they paid for the hog. They sell the other hog for 80% of the price they paid for it. Do they make a profit or a loss overall, and how much, in dollars, is that profit or loss?
Answer: Loss of \$6.
Solution: The price of the first hog was $(\frac{4}{5})120 = 4 \times 24 = 96$. There was a profit of \$120-\$96=\$24 on that hog. The price of the second hog was $(\frac{5}{4})120 = 5 \times 30 = 150$. There was a loss of \$150-\$120=\$30 on that hog. There was an overall loss of $\$30 - 24 = \$6$.

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5. A box of 500 balls contains balls numbered $1, 2, 3, \ldots 100$ in each of five different colors. Without ever looking at any of the balls, you are to choose balls at random from the box and put them into a bag. If you must be sure that when you finish, the bag contains at least one set of five balls with identical numbers, then what is the smallest number of balls that you can put in the bag?
Answer: 401
Solution: If you happen to put four of the balls with each number in the bag, you will have 400 balls but will NOT have any set of five balls with identical numbers in the bag. Therefore, you need to choose at least 401 balls. On the other hand, if you do choose 401 balls, then you have left behind 99 balls and there must be at least one number that has not been left behind! All five balls with that number will be in your bag. So 401 balls will do, but no smaller number will do.

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6. Tutu, Jada, and Faith eat lunch together. Tutu contributes 9 sausages, Jada contributes 8 sausages, and the three girls divide the sausages equally. Faith has brought no food, but gives the other two girls 17 wupiupi coins in exchange for her share of the sausages. How many of the coins should Tutu get?
Answer: 10 coins go to Tutu (and the other 7 coins go to Jada).
Solution: Each girl will get $\frac{17}{3}$ sausages, so Faith is trading 17 wupiupi for $\frac{17}{3}$ sausages, or 3 coins per sausage. Tutu will trade $9 - \frac{17}{3}$ sausages to Faith,
or $\frac{10}{3}$ sausages and will get 10 coins. (Jada will get the other 7 coins in exchange for $8 - \frac{17}{3}$ or $\frac{7}{3}$ sausages.)

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7. What is the first time after midnight at which the hour hand and minute hand on an ordinary clock face are perpendicular to one another? Express the time in the format Hour, Minute, Second, with your answer rounded to the nearest second. Assume the clock is a 12 hour clock with hands that move at uniform speeds.
Answer: 12 hours,16 minutes, 22 seconds (Or 0 hours, 16 minutes, 22 seconds)
Solution: Let m be the number of minutes past midnight. The hour hand will point at the $\frac{m}{60} \times 5$ minute mark, or $\frac{m}{12}$ minutes. The minute hand will be at m. The minute hand will be 90 degrees ahead of the hour hand when it is exactly 15 minute marks ahead. Set $m = 15 + \frac{m}{12}$. Solving, we get $\frac{11m}{12} = 15$ or $m = 12 \times \frac{15}{11} = \frac{180}{11} = 16\frac{4}{11}$. To put the answer in the form required, we must convert $\frac{4}{11}$ minutes to the nearest second. $\frac{4}{11}$ minutes = $\frac{4}{11} \times 60$ seconds $= \frac{240}{11}$ seconds $= 21\frac{9}{11}$ seconds rounded to the nearest second is 22 seconds. Therefore, the time is 16 minutes and 22 seconds past midnight. (It is acceptable to give the hour as either 0 or 12.)

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8. How many integers greater than 0 and less than $100,000$ are palindromes? An integer is a palindrome if its digits are the same when read left to right and right to left. For instance, 2134312 and 353 are palindromes; so are $1001, 99, 5$, and 1. Reminder: do not count the number 0.
Answer: 1098
Solution: Count the number of 1 digit palindromes, the number of 2 digit palindromes, and so on. All 9 positive single digits are palindromes. A 2 digit palindrome is of the form aa, and there are 9 possibilities for a. There are 9 two digit palindromes- $11, 22, 33$, etc. Three digit palindromes are of the form aba. The a cannot be 0, so there are 9 possibilities for a. All 10 digits are possibilities for b. Therefore there are $9 \times 10 = 90$ three digit palindromes. The four digit palindromes are of the form $abba$, where again a cannot be 0, so there are also 90 four digit palindromes. The five digit palindromes are of the form $abcba$, with 9 possibilities for a and 10 possibilities for each of b and c. There are thus $9 \times 10 \times 10 = 900$ five digit palindromes. In all, there are $9 + 9 + 90 + 90 + 900 = 1098$ positive palindromes less than $100,000$.

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9. A polynomial $P(x)$ satisfies the equation

$$P(P(x) - 1) = 1 + x^{16}.$$

What is $P(2)$? (The expression $P(P(x) - 1)$ on the left side of the equation means "plug $P(x) - 1$ into P." The parentheses in this case do not indicate multiplication.)
Answer: 17
Solution: Give the name $Q(x)$ to the expression $P(x) - 1$. The equation says that $Q(Q(x)) = x^{16}$. Clearly $Q(x) = x^4$ and $P(x) = Q(x) + 1 = x^4 + 1$. (It is a good idea to check a problem like this! That is, verify that this P satisfies the given equation.) Compute $P(2) = 2^4 + 1 = 16 + 1 = 17$.

10. How many different case-sensitive passwords can be created with at most 8 keystrokes, if each keystroke may touch either the "caps lock" key or any of the 10 alphabetic keys on the top row of the keyboard: Q W E R T Y U I O P ? Assume that password entry always begins with "caps lock" in lowercase mode, and assume that a password must contain at least one letter. Tapping the "caps lock" key toggles the mode of the keyboard between lowercase and uppercase. Assume that holding a key down does not produce multiple copies of a letter; that is, in the password field, holding a key down has no effect.
Answer: $204,768,420$
Solution: Count separately the number of 8 letter passwords, the number of 7 letter passwords, and so on.
If a password is 8 letters long, then caps lock is not used at all. There are 10^8 possible choices- 10 possible letters for each of 8 slots.
If a password is 7 letters long, the there is at most 1 touch of caps lock. There are 10^7 passwords with 7 letters and no caps lock. For each of these, the caps lock can be touched before any of the 7 letters, so there are 7×10^7 passwords with 7 letters and one touch of caps lock.
If a password is 6 letters long, then it can have $0, 1$ or 2 touches of the caps lock. There are 10^6 passwords with 6 letters and no touches of caps lock. There are 6×10^6 passwords with 6 letters and one touch of the caps lock. If two touches of caps lock are done consecutively, the password that results will be the same as what results with no touch of caps lock in that place. Such passwords are being counted already. So we need count only passwords in which caps lock is not hit twice consecutively. We can choose any two of the 6 letters to place a caps lock in front of, so the number of passwords with two touches of caps lock and 6 letters is $\binom{6}{2} \times 10^6 = 15 \times 10^6$. Continue on in this way, considering passwords of $5, 4, 3, 2,$ and 1 letters. The Pascal triangle can be helpful for quick computation of the choose numbers.

END OF CONTEST