Twenty-fifth Annual UNC Math Contest **Final Round** **January 21, 2017**

Rules: Three hours; no electronic devices.
The positive integers are 1, 2, 3, 4, ...
A prime is an integer strictly greater than one that is evenly divisible by no integers other than itself and 1. The primes are 2, 3, 5, 7, 11, 13, 17, ...

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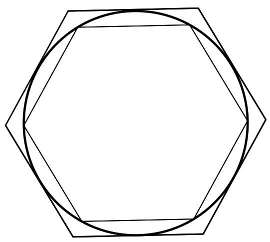
1. A circle has radius 24, a second circle has radius 15, and the centers of the two circles are 52 units apart. A line tangent to both circles crosses the line connecting the two centers at a point P between the two centers. How much farther is P from the center of the bigger circle than it is from the center of the smaller circle?



2. Find the ratio of the area of a regular hexagon circumscribed around a circle to the area of a regular hexagon inscribed inside the same circle. (A polygon is called *regular* if all its sides are the same length and all its corner angles have the same measure. A *hexagon* is a polygon with six sides.)

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3. Prime mates Find the largest 9 digit integer in which no two digits are the same and the sum of each pair of adjacent digits is prime. That is, the sum of the first two digits is prime, the sum of the second and third digits is prime, the sum of the third and fourth digits is prime, and so on.

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4. Monkey business Harold writes an integer; its right-most digit is 4. When Curious George moves that digit to the far left, the new number is four times the integer that Harold wrote. What is the smallest possible positive integer that Harold could have written?

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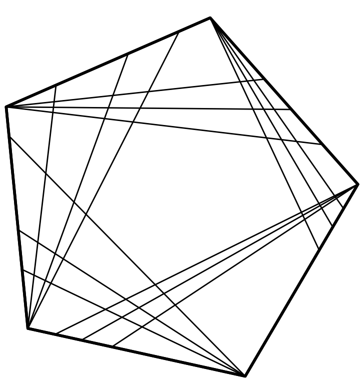
5. Double encryption (a) Find a substitution code on the seven letters A, B, C, D, E, F, and G that has the property that if you apply it twice in a row (that is, encrypt the encryption), the message ABCDEFG becomes ECBFAGD. Describe your answer by giving the message that results when encryption is applied once to the message ABCDEFG.

(b) Find another such code, if there is one.



6. The spider′s divider On a regular pentagon, a spider forms segments that connect one endpoint of each side to n different non-vertex points on the side adjacent to the other endpoint of that side, going around clockwise, as shown. Into how many non-overlapping regions do the segments divide the pentagon? Your answer should be a formula involving n. (In the diagram, $n = 3$ and the pentagon is divided into 61 regions.)

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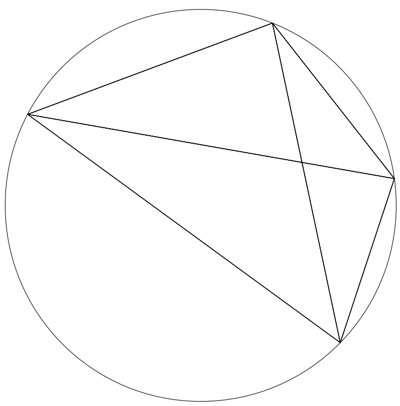
7. A box of 48 balls contains balls numbered 1, 2, 3, . . ., 12 in each of four different colors. Without ever looking at any of the balls, you choose balls at random from the box and put them in a bag.
(a) If you must be sure that when you finish, the bag contains at least one set of five balls whose numbers are consecutive, then what is the smallest number of balls you can put in the bag? (For example, a set of balls, in any combination of colors, with numbers 3, 4, 5, 6, and 7 is a set of five whose numbers are consecutive.)

(b) If instead you must be sure that the bag contains at least one set of five balls all in the same color and with consecutive numbers, then what is the smallest number of balls you can put in the bag? Remember to justify answers for maximum credit.

8. For what integer n does $x^2 - x + n$ divide into $x^{13} - 233x - 144$ with no remainder? That is, for what integer n is the first polynomial a factor of the second one? As always, justify your answer.



9. Suppose n points on the circumference of a circle are joined by straight line segments in all possible ways and that no point that is not one of the original n points is contained in more than two of the segments. How many triangles are formed by the segments? Count all triangles whose sides lie along the segments, including triangles that overlap with other triangles. For example, for $n = 3$ there is one triangle and for $n = 4$ (shown in the diagram) there are 8 triangles.

10. Powerless progressions Find an infinite sequence of integers $a_1, a_2, a_3, \ldots$ that has all of these properties:
(1) $a_n = c + dn$ with c and d the same for all $n = 1, 2, 3, \ldots$
(2) c and d are positive integers, and
(3) no number in the sequence is the r^{th} power of any integer, for any power $r = 2, 3, 4, \ldots$
Reminder: Justify answers. In particular, for maximum credit, make it clear in your presentation that your sequence possesses the third property.

11. Divide and conquer (a) How many different factorizations are there of 4096 (which is the twelfth power of 2) in which each factor is either a square or a cube (or both) of an integer and each factor is greater than one? Regard $4 \times 4 \times 4 \times 8 \times 8$ and $4 \times 8 \times 4 \times 8 \times 4$ as the same factorization: the order in which the factors are written does not matter. Regard the number itself, 4096, as one of the factorizations.

(b) How many different factorizations are there of 46,656 as a product of factors in which each factor is either a square or a cube (or both) of an integer and each factor is greater than one? As before, the order in which the factors is written does not matter, and the number itself counts as a factorization. Note that $46,656 = 2^6 \times 3^6$.

END OF CONTEST