University of Northern Colorado Mathematics Contest 2016-2017

Solutions of Final Round

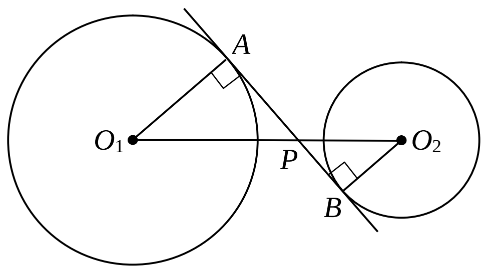
1. A circle has radius 24, a second circle has radius 15, and the centers of the two circles are 52 units apart. A line tangent to both circles crosses the line connecting the two centers at a point P between the two centers. How much farther is P from the center of the bigger circle than it is from the center of the smaller circle?

Answer: 12

Solution:

Draw the diagram.

Let O_1 and O_2 be the centers of the bigger circle and the smaller circle respectively. Let A and B be the tangent points of the tangent line to the two circles respectively. Draw O_1A and O_2B. Then $O_1A \perp AB$ and $O_2B \perp AB$.

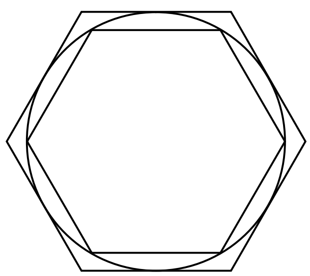


Obviously, $\Delta O_1AP \sim \Delta O_2BP$. We have $\frac{O_1P}{O_2P} = \frac{O_1A}{O_2B}$. That is, $\frac{O_1P}{O_2P} = \frac{24}{15} = \frac{8}{5}$.

Note that $O_1P + O_2P = 52$. So $O_1P = 52 \cdot \frac{8}{8+5} = 32$ and $O_2P = 52 - 32 = 20$.

The answer is $32 - 20 = 12$.

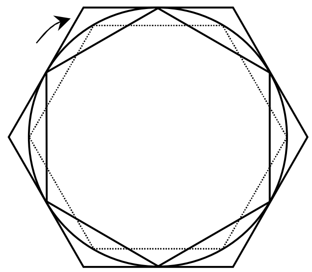
2. Find the ratio of the area of a regular hexagon circumscribed around a circle to the area of a regular hexagon inscribed inside the same circle. (A polygon is called regular if all its sides are the same length and all its corner angles have the same measure. A hexagon is a polygon with six sides.)



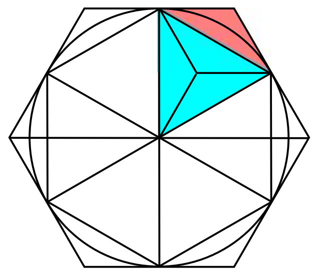
Answer: $\frac{4}{3}$

Solution 1:

Rotate the inscribed triangle clockwise by $30°$.

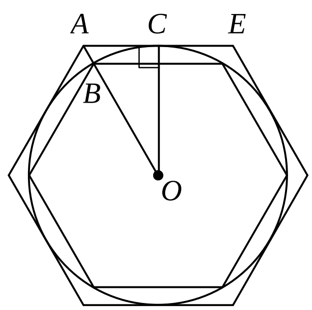


Let us shade the $\frac{1}{6}$ of the whole shape as shown. We see that the answer is $\frac{4}{3}$.



Solution 2:

Let O be the center of the circle.



Let *A* be a vertex of the larger hexagon and *B* be a vertex of the smaller hexagon. We may rotate one hexagon such that *A*, *B*, and *O* are in a line. Let *AE* be a side of the larger hexagon.

Draw $OC \perp AE$ with *C* on *AE*. Obviously, *C* is a tangent point. So *ACO* is a 30°-60°-90° triangle. We have $\frac{OA}{OC} = \frac{2}{\sqrt{3}}$. Since $OB = OC$, $\frac{OA}{OB} = \frac{2}{\sqrt{3}}$.

Note that regular hexagons are naturally similar. The area ratio of the larger hexagon to the smaller is the square of the ratio of the corresponding lengths:

$$\left(\frac{OA}{OB}\right)^2 = \left(\frac{2}{\sqrt{3}}\right)^2 = \frac{4}{3}.$$

The answer is $\frac{4}{3}$.

3. Prime mates Find the largest 9-digit integer in which no two digits are the same and the sum of each pair of adjacent digits is prime. That is, the sum of the first two digits is prime, the sum of the second and third digits is prime, the sum of the third and fourth digits is prime, and so on.

Answer: 985674321

Solution:

We start from 987654321.

It is good that $9+8=17$ is a prime, but $8+7=15$ is not.

We have to replace at least one of 8 and 7.

Since we want the number to be greatest, we replace 7 with some digit. We can replace 7 only with an odd digit. The next greatest odd digit is 5. Let us switch 7 and 5. We have 985674321. Now the number satisfies all conditions.

The answer is 985674321.

4. **Monkey business** Harold writes an integer; its right-most digit is 4. When Curious George moves that digit to the far left, the new number is four times the integer that Harold wrote. What is the smallest possible positive integer that Harold could have written?

Answer: 102564

Solution 1:

Let *N* be an *n*-digit number. Let Harold's number be $\overline{N4}$ formed by placing 4 at the right to *N*.

The new number after George's change is $\overline{4N}$. We have $\overline{4N} = 4\cdot\overline{N4}$. That is,

$$4\cdot 10^n + N = 4\cdot(10N+4).$$

We have

$$N = \frac{4\cdot(10^n - 4)}{39}.$$

We must have $10^n - 4$ to be divisible by 39. That is, $\overset{|\leftarrow n-1\text{ 9's}\rightarrow|}{99\ \cdots\ 9}6$ is divisible by 39.

Do the division. It is easy to see that 99996 is divisible by 6 and it is the smallest possible. With $99996 \div 4 = 2564$, the smallest value for *N* is $4\cdot 2564 = 10256$.

Therefore, Harold's smallest possible number is 102564.

Solution 2:

Let Harold's number be $\overline{a_n\cdots a_2a_14}$ where $a_n, \cdots, a_2$, a_1 are digits with $a_n \ge 1$. Then the new number is $\overline{4a_n\cdots a_2a_1}$.

Note that $4\times\overline{a_n\cdots a_2a_14} = \overline{4a_n\cdots a_2a_1}$. So $a_1 = 6$.

We see $4\times\overline{a_n\cdots a_2 64}=\overline{4a_n\cdots a_2 6}$. We see $a_2=5$. We have $4\times\overline{a_n\cdots a_3 564}=\overline{4a_n\cdots a_3 56}$.

Then $a_3=2 \;\Rightarrow\; 4\times\overline{a_n\cdots a_4 2564}=\overline{4a_n\cdots a_4 256} \;\Rightarrow\; a_4=0 \;\Rightarrow$

$4\times\overline{a_n\cdots a_5 02564}=\overline{4a_n\cdots a_5 0256} \;\Rightarrow\; a_5=1 \;\Rightarrow\; 4\times\overline{a_n\cdots a_6 102564}=\overline{4a_n\cdots a_6 10256}$.

Now we must have $a_6=4$. Since $a_6=4$ matches the leading digit of the new number, we stop here to obtain the smallest possible number.

The multiplication is $4\times 102564=410256$.

Therefore, Harold's smallest possible number is 102564.

5. **Double encryption** (a) Find a substitution code on the seven letters *A*, *B*, *C*, *D*, *E*, *F*, and G that has the property that if you apply it twice in a row (that is, encrypt the encryption), the message *ABCDEFG* becomes *ECBFAGD*. Describe your answer by giving the message that results when encryption is applied once to the message *ABCDEFG*.

(b) Find another such code, if there is one.

Answer: *BEAGCDF* and *CAEGBDF*

Solution:

We use 1, 2, $\cdots$ to replace *A*, *B*, $\cdots$.

After two operations we see

	1	2	3	4	5	6	7
after 2 operations	5	3	2	6	1	7	4

The rule for the combined two operations is:

$$1\to 5\to 1,$$

$$2\to 3\to 2,$$

$$4\to 6\to 7\to 4.$$

Look at the group with 4-6-7.

Since there are only three numbers, it is easy to see

	4	6	7
after 1st operation	7	4	6
after 2nd operation	6	7	4

For the group with 1-5 and the group with 2-3 we have two choices:

	1	5	2	3			1	5	2	3
after 1st operation	2	3	5	1		after 1st operation	3	2	1	5
after 2nd operation	5	1	3	2	or	after 2nd operation	5	1	3	2

We have two solutions:

	1	5	2	3	4	6	7
after one operation	2	3	5	1	7	4	6

and

	1	5	2	3	4	6	7
after one operation	3	2	1	5	7	4	6

Make them in the right order:

	1	2	3	4	5	6	7
after one operation	2	5	1	7	3	4	6

and

	1	2	3	4	5	6	7
after one operation	3	1	5	7	2	4	6

Translate 1, 2, $\cdots$ back to *A*, *B*, $\cdots$. The two solutions are

	A	*B*	*C*	*D*	*E*	*F*	*G*
after one operation	*B*	*E*	*A*	*G*	*C*	*D*	*F*

and

	A	*B*	*C*	*D*	*E*	*F*	*G*
after one operation	*C*	*A*	*E*	*G*	*B*	*D*	*F*

We have an elegant way to decode if there are more numbers.

For this example, look at the group with 4-6-7.

	4	6	7
after 1st operation	?	?	?
after 2nd operation	6	7	4

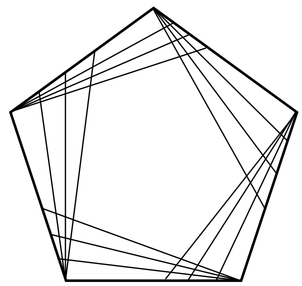
The key observation is that the sequence goes back to 4-6-7 after 3 operations.

	4	6	7
after 1st operation	?	?	?
after 2nd operation	6	7	4
after 3rd operation	4	6	7

We find the one-operation rule from the result of the 2nd operation to the result of the 3rd operation

$$4 \to 7 \to 6 \to 4\,.$$

6. **The spider's divider** On a regular pentagon, a spider forms segments that connect one endpoint of each side to *n* different non-vertex points on the side adjacent to the other endpoint of that side, going around clockwise, as shown. Into how many non-overlapping regions do the segments divide the pentagon? Your answer should be a formula involving *n*. (In the diagram, $n = 3$ and the pentagon is divided into 61 regions.)



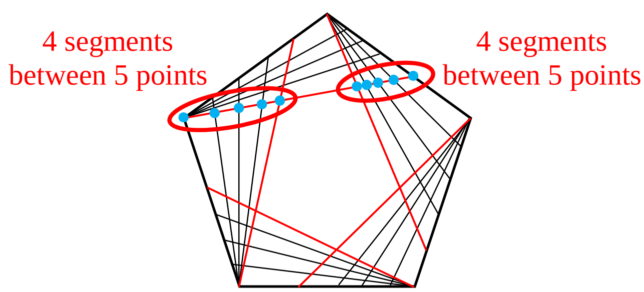
Answer: $5n^2 + 5n + 1$

Solution:

At the beginning we have one region – the pentagon. From every vertex draw the 1st line. It will increase the number of regions by 10. By drawing the second line from every vertex, the number of regions will be increased by 20. By drawing the third line from every vertex, the number of regions will be increased by 30. In general, by drawing the n^{th} line from every vertex, the number of regions will be increased by 10*n*. So the number of regions after *n* lines are drawn from every vertex is

$$1 + 10(1 + 2 + \cdots + n) = 1 + 5n(n+1) = 5n^2 + 5n + 1.$$

How do we see that the number of regions is increased by $10n$ when we draw the n^{th} line from every vertex?



We generate $n + n = 2n$ (4 + 4 = 8 in the diagram) segments in each newly drawn line when we draw the n^{th} line (4^{th} line in the diagram) from every vertex. Each segment divides a region into two regions making one more region. So the total number of more regions produced by drawing the n^{th} line is $2n \cdot 5 = 10n$.

7. A box of 48 balls contains balls numbered 1, 2, 3, . . ., 12 in each of four different colors. Without ever looking at any of the balls, you choose balls at random from the box and put them in a bag.

   (a) If you must be sure that when you finish, the bag contains at least one set of five balls whose numbers are consecutive, then what is the smallest number of balls you can put in the bag? (For example, a set of balls, in any combination of colors, with numbers 3, 4, 5, 6, and 7 is a set of five whose numbers are consecutive.)

(b) If instead you must be sure that the bag contains at least one set of five balls all in the same color and with consecutive numbers, then what is the smallest number of balls you can put in the bag? Remember to justify answers for maximum credit.

Answer: (a) 41; (b) 41

Solution:

The solution is for both (a) and (b).

You can put all the balls except the balls with numbers 5 and 10 in your bag and you will have 40 balls with no runs of five consecutive numbers. Therefore, the answer is larger than 40.

Now we show that 41 is enough. Suppose that you have 41 balls. At least in one color you have 11 or more balls by the pigeonhole principle. In that color, you will have 5 balls with five consecutive numbers.

The answer is 41 for both.

8. For what integer *n* does $x^2 - x + n$ divide into $x^{13} - 233x - 144$ with no remainder? That is, for what integer *n* is the first polynomial a factor of the second one? As always, justify your answer.

Answer: -1

Solution 1:

Note that 144 and 233 are the Fibonacci numbers, and note that $x^2 - x - 1$ has the golden ratio $\frac{1+\sqrt{5}}{2}$ as a root. We know how the Fibonacci sequence is related to the golden ratio. We guess $n = -1$. By knowing the answer, we can easily achieve the factorization:

$$\begin{aligned}&\left(x^2 - x - 1\right)\left(x^{11} + x^{10} + 2x^9 + 3x^8 + 5x^7 + 8x^6 + 13x^5 + 21x^4 + 34x^3 + 55x^2 + 89x + 144\right)\\ &\quad = x^{13} - 233x - 144\end{aligned}$$

The answer is -1.

Solution 2:

Let $x = 0$. We have $n \mid 144$.

Let $x = 1$. We have $n \mid 376$.

The greatest common factor of 144 and 376 is 8. Then $n \mid 8$.

So *n* can be ± 1, ± 2, ± 4, ± 8.

Let $x = -1$. We have $n + 2 \mid 88$. It leaves 2, -1, -4 for *n*.

Let $x = 2$. We have $n + 2 \mid 7582$. Now we have -1, -4 for *n*.

Let $x = 4$. We have $n + 12 \mid 67107788$. Only -1 is possible for *n*

The answer is -1.

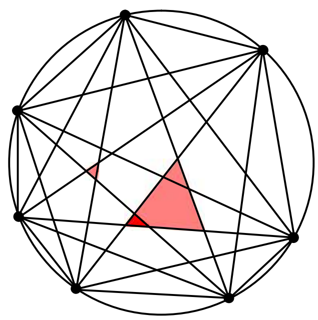
9. Suppose n points on the circumference of a circle are joined by straight line segments in all possible ways and that no point that is not one of the original n points is contained in more than two of the segments. How many triangles are formed by the segments? Count all triangles whose sides lie along the segments, including triangles that overlap with other triangles. For example, for $n=3$ there is one triangle and for $n=4$ (shown in the diagram) there are 8 triangles.

Answer: $\binom{n}{6}+5\binom{n}{5}+4\binom{n}{4}+\binom{n}{3}$

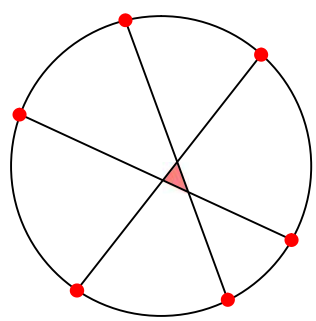
Solution to (a):

There are four kinds of triangles.

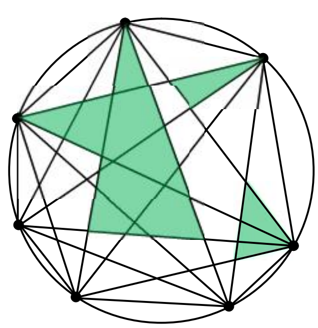
Kind 1: all three vertices of a triangle are inside the circle. I shade several of this kind in the following figure.



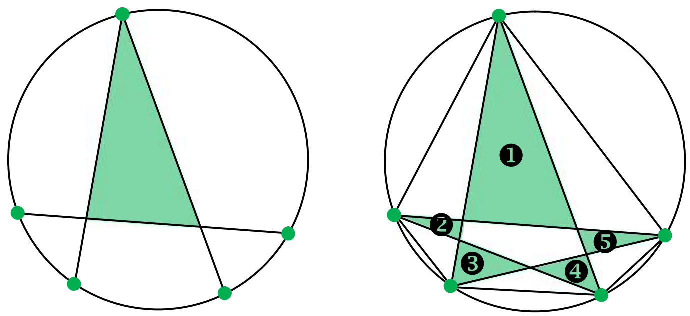
One triangle of kind 1 is determined by 6 points on the circle, and 6 points determine only one triangle. So there are $\binom{n}{6}$ triangles of this kind.



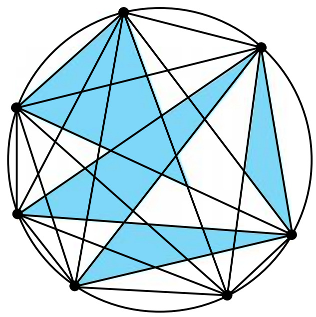
Kind 2: exactly one vertex of a triangle is on the circle. I shade several of this kind as shown.



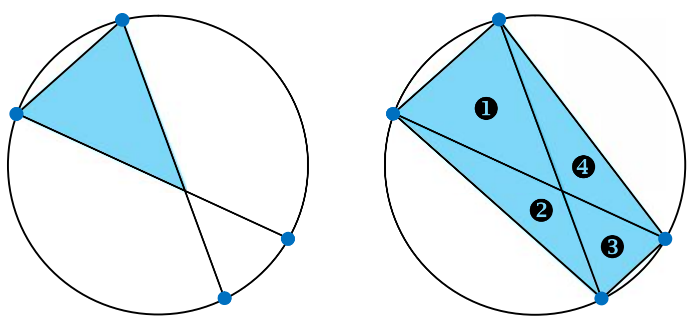
Five points are involved in one triangle of this kind. However, when we choose 5 points, we can make 5 triangles of this kind. So there are $5\binom{n}{5}$ triangles of this kind.



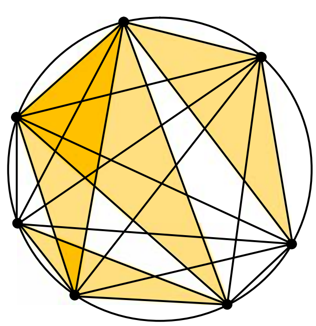
Kind 3: exactly two vertices of a triangle are on the circle. I shade several of this kind:



Four points are involved in one triangle of this kind. However, when we choose 4 points, we can make 4 triangles of this kind. So there are $4\binom{n}{4}$ triangles of this kind.



Kind 4: all three vertices of a triangle are on the circle. I shade several of this kind:



Obviously there are $\binom{n}{3}$ triangles of this kind.

The total number of triangles is

$$\binom{n}{6}+5\binom{n}{5}+4\binom{n}{4}+\binom{n}{3}.$$

10. **Powerless progressions** Find an infinite sequence of integers $a_1, a_2, a_3, \cdots$ that has all of these properties:

(1) $a_n = c + dn$ with c and d the same for all $n = 1, 2, \cdots$

(2) c and d are positive integers, and

(3) no number in the sequence is the r^{th} power of any integer, for any power $r = 2, 3, \cdots$.

Reminder: Justify answers. In particular, for maximum credit, make it clear in your presentation that your sequence possesses the third property.

Answer: $2, 6, 10, \cdots$ (many possible different sequences)

Solution:

To avoid the powers of odd numbers, we will build a sequence with even number. Let $2k$ be an even number. Its r^{th} power $(2k)^r = 2^r k^r$ ($r \ge 2$) is a multiple of 4. So to avoid the powers of even numbers, we build a sequence without a multiple of 4.

Then let $c = 2$ and $d = 4$.

We have a sequence $2, 6, 10, \cdots$ satisfying all conditions.

11. **Divide and conquer** (a) How many different factorizations are there of 4096 (which is the twelfth power of 2) in which each factor is either a square or a cube (or both) of an integer and each factor is greater than one? Regard $4\times4\times4\times8\times8$ and $4\times8\times4\times8\times4$ as the same factorization: the order in which the factors are written does not matter. Regard the number itself, 4096, as one of the factorizations.

(b) How many different factorizations are there of 46,656 as a product of factors in which each factor is either a square or a cube (or both) of an integer and each factor is greater than one? As before, the order in which the factors is written does not matter, and the number itself counts as a factorization. Note that $46{,}656 = 2^6 \cdot 3^6$.

Answer: (a) 16; (b) 42

Solution to (a):

We need to find the number of ways to partition 12 into a non-decreasing sequence consisting of multiples of 2's or 3's with one or more terms.

Let us list the ways systematically according to the number of terms:

2	2	2	2	2	2
2	2	2	2	4	

2	2	2	3	3
2	2	2	6	
2	2	4	4	
2	3	3	4	
3	3	3	3	
2	2	8		
2	4	6		
3	3	6		
4	4	4		
2	10			
3	9			
4	8			
6	6			
12				

There are 16 ways.

Solution to (b):

There are 4 ways to express 2^6 as a product of one or more squares and/or cubes:

$$2^2 \cdot 2^2 \cdot 2^2$$

$$2^2 \cdot 2^4$$

$$2^3 \cdot 2^3$$

$$2^6$$

and there are 4 ways to express 3^6 as a product of one or more squares and/or cubes:

$$3^2 \cdot 3^2 \cdot 3^2$$

$$3^2 \cdot 3^4$$

$$3^3 \cdot 3^3$$

$$3^6$$

So there are $4 \cdot 4 = 16$ ways to put an expression from the first list and an expression from the second list together without combing any terms.

Now we take a look at how many ways there are to combine the terms for each possible combination of the two expressions.

(1) $2^2 \cdot 2^2 \cdot 2^2$

(a) $2^2 \cdot 2^2 \cdot 2^2 \quad \cdot \quad 3^2 \cdot 3^2 \cdot 3^2$

There are three ways to combine terms:

$$(2\cdot 3)^2\cdot 2^2\cdot 2^2\cdot 3^2\cdot 3^2$$

$$(2\cdot 3)^2\cdot (2\cdot 3)^2\cdot 2^2\cdot 3^2$$

$$(2\cdot 3)^2\cdot (2\cdot 3)^2\cdot (2\cdot 3)^2$$

(b) $2^2\cdot 2^2\cdot 2^2 \quad\cdot\quad 3^2\cdot 3^4$

There are three ways to combine terms:

$$(2\cdot 3)^2\cdot 2^2\cdot 2^2\cdot 3^4$$

$$\left(2\cdot 3^2\right)^2\cdot 2^2\cdot 2^2\cdot 3^2$$

$$(2\cdot 3)^2\cdot \left(2\cdot 3^2\right)^2\cdot 2^2$$

(c) $2^2\cdot 2^2\cdot 2^2 \quad\cdot\quad 3^3\cdot 3^3$

There is no way to combine terms.

(d) $2^2\cdot 2^2\cdot 2^2 \quad\cdot\quad 3^6$

There is one way to combine terms: $\left(2\cdot 3^3\right)^2\cdot 2^2\cdot 2^2$.

In this case there are $3+3+0+1=7$ ways.

(2) $2^2\cdot 2^4$

(a) $2^2\cdot 2^4 \quad\cdot\quad 3^2\cdot 3^2\cdot 3^2$

There are three ways to combine terms:

$$2^4\cdot (2\cdot 3)^2\cdot 3^2\cdot 3^2$$

$$2^2\cdot \left(2^2\cdot 3\right)^2\cdot 3^2\cdot 3^2$$

$$(2\cdot 3)^2\cdot \left(2^2\cdot 3\right)^2\cdot 3^2$$

(b) $2^2\cdot 2^4 \quad\cdot\quad 3^2\cdot 3^4$

There are six ways to combine terms:

$$(2\cdot 3)^2\cdot 2^4\cdot 3^4$$

$$\left(2\cdot 3^2\right)^2\cdot 2^4\cdot 3^2$$

$$2^2\cdot \left(2^2\cdot 3\right)^2\cdot 3^4$$

$$2^2\cdot \left(2^2\cdot 3^2\right)^2\cdot 3^2$$

$$(2\cdot 3)^2\cdot \left(2^2\cdot 3^2\right)^2$$

$$\left(2\cdot 3^2\right)^2\cdot \left(2^2\cdot 3\right)^2$$

(c) $2^2 \cdot 2^4 \quad \cdot \quad 3^3 \cdot 3^3$

There is no way to combine terms.

(d) $2^2 \cdot 2^4 \quad \cdot \quad 3^6$

There are two ways to combine terms:

$$\left(2 \cdot 3^3\right)^2 \cdot 2^4.$$

$$2^2 \cdot \left(2^2 \cdot 3^3\right)^2.$$

In this case there are $3+6+0+2=11$ ways.

(3) $2^3 \cdot 2^3$

(a) $2^3 \cdot 2^3 \quad \cdot \quad 3^2 \cdot 3^2 \cdot 3^2$

There is no way to combine terms.

(b) $2^3 \cdot 2^3 \quad \cdot \quad 3^2 \cdot 3^4$

There is no way to combine terms.

(c) $2^3 \cdot 2^3 \quad \cdot \quad 3^3 \cdot 3^3$

There are two ways to combine terms:

$$\left(2 \cdot 3\right)^3 \cdot 2^3 \cdot 3^3.$$

$$\left(2 \cdot 3\right)^3 \cdot \left(2 \cdot 3\right)^3.$$

(d) $2^3 \cdot 2^3 \quad \cdot \quad 3^6$

There is one way to combine terms: $\left(2 \cdot 3^2\right)^3 \cdot 2^3$.

In this case there are $0+0+2+1=3$ ways.

(4) 2^6

(a) $2^6 \quad \cdot \quad 3^2 \cdot 3^2 \cdot 3^2$

There is one way to combine terms: $\left(2^3 \cdot 3\right)^2 \cdot 3^2 \cdot 3^2 \cdot 3^2$.

(b) $2^6 \quad \cdot \quad 3^2 \cdot 3^4$

There are two ways to combine terms:

$$\left(2^3 \cdot 3\right)^2 \cdot 3^4$$

$$3^2 \cdot \left(2^3 \cdot 3^2\right)^2$$

(c) $2^6 \quad \cdot \quad 3^3 \cdot 3^3$

There is one way to combine terms: $\left(2^2 \cdot 3\right)^3 \cdot 3^3$.

(d) $2^6 \quad \cdot \quad 3^6$

There is one way to combine terms: $(2 \cdot 3)^6$.

In this case there are $1+2+1+1=5$ ways.

The total number of ways is

$$16+7+11+3+5=42.$$