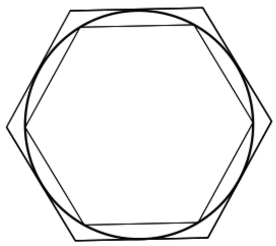
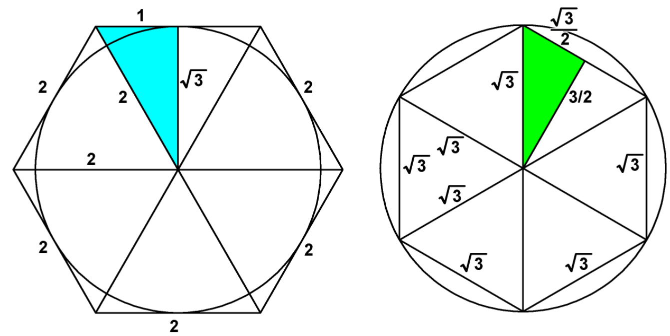


2. Find the ratio of the area of a regular hexagon circumscribed around a circle to the area of a regular hexagon inscribed inside the same circle. (A polygon is called *regular* if all its sides are the same length and all its corner angles have the same measure. A *hexagon* is a polygon with six sides.)

Solution: Compute the areas. Let's start with the circumscribed (larger) hexagon. In the diagram below, I divide the larger hexagon into six equilateral triangles. And I divide one of those equilateral triangles into two 30-60-90 triangles. Since we are asked for the ratio of areas, we can choose one arbitrary and convenient dimension. I choose the side length of the equilateral triangles is 2. For any 30-60-90 triangle, the sides are always in the ratio $1:\sqrt{3}:2$. The short leg is half the hypotenuse. And the long leg is $\sqrt{3}$ times the short side, or $\sqrt{3}$. (See the blue triangle in the left-hand figure.) The area of this triangle is $\frac{1}{2}b\cdot h=\frac{1}{2}\cdot 1\cdot\sqrt{3}=\frac{\sqrt{3}}{2}$. The entire area of the circumscribed hexagon is the same as 12 blue triangles.



Now, let's look at the inscribed (smaller) hexagon. Since the circle is the same size, we are stuck with the dimensions we chose for the circumscribed hexagon. The radius of the circle is $\sqrt{3}$. We divide the smaller hexagon into six equilateral triangles, with sides of length $\sqrt{3}$. We divide one of those equilateral triangles into two 30-60-90 triangles. The hypotenuse has length $\sqrt{3}$, so the short side has half that length, or $\frac{\sqrt{3}}{2}$. The long leg is $\sqrt{3}$ times the short leg, or $\frac{3}{2}$. (See the green triangle in the right-hand figure.) The area of this triangle is $\frac{1}{2}b\cdot h=\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\cdot\frac{3}{2}=\frac{3\sqrt{3}}{8}$. The entire area of the inscribed hexagon is the same as 12 green triangles.

Finally, the question asks for the ratio of the areas of the circumscribed hexagon to the inscribed hexagon:

$$\frac{A_{\text{circumscribed}}}{A_{\text{inscribed}}}=\frac{12\times\frac{\sqrt{3}}{2}}{12\times\frac{3\sqrt{3}}{8}}=\frac{\frac{1}{2}}{\frac{3}{8}}=\frac{1}{2}\times\frac{8}{3}=\boxed{\frac{4}{3}}$$

4. Monkey business Harold writes an integer; its right-most digit is 4. When Curious George moves that digit to the far left, the new number is four times the integer that Harold wrote. What is the smallest possible positive integer that Harold could have written?

Solultion: We can simply write the answer from right-to-left, by performing a standard long-hand multiplication by 4, transcribing digits from the product to the multiplicand, as they are found.

We know the units digit of the unknown original number is 4. We know we are seeking another number that is 4 times the unknown original number.

Step 1. 4 times 4 is 16. Write the 6. Carry the 1. Transfer the product digit, 6, to the next step.

```
□□□□□□1  ← carries
□□□□□□□4 ← multiplicand
      ×4 ← multiplier
--------
□□□□□□□6 ← product
```

Step 2. Transfer the digit, 6, from the previous step to the multiplicand. 6 times 4 is 24. Plus the carry (1) is 25. Write the 5. Carry the 2. Transfer the product digit, 5, to the next step.

```
□□□□□21  ← carries
□□□□□□64 ← multiplicand
      ×4 ← multiplier
--------
□□□□□□56 ← product
```

Step 3. Transfer the digit, 5, from the previous step to the multiplicand. 5 times 4 is 20. Plus the carry (2) is 22. Write the 2. Carry the 2. Transfer the product digit, 2, to the next step.

```
□□□□221  ← carries
□□□□□564 ← multiplicand
      ×4 ← multiplier
--------
□□□□□256 ← product
```

Step 4. Transfer the digit, 2, from the previous step to the multiplicand. 2 times 4 is 8. Plus the carry (2) is 10. Write the 0. Carry the 1. Transfer the product digit, 0, to the next step.

```
□□□1221  ← carries
□□□□2564 ← multiplicand
      ×4 ← multiplier
--------
□□□□0256 ← product
```

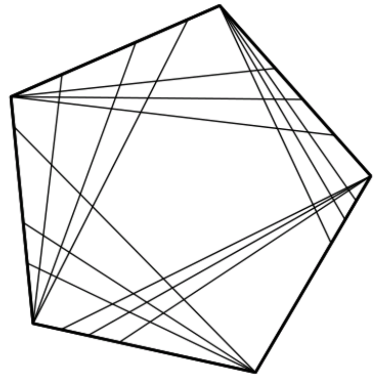
Step 5. Transfer the digit, 0, from the previous step to the multiplicand. 0 times 4 is 0. Plus the carry (1) is 1. Write the 1. Carry a zero. Transfer the product digit, 1, to the next step.

```
□□01221  ← carries
□□□02564 ← multiplicand
      ×4 ← multiplier
--------
□□□10256 ← product
```

Step 6. Transfer the digit, 1, from the previous step to the multiplicand. 1 times 4 is 4. Plus the carry (0) is 4. Write the 4. Carry a zero. Stop. Or transfer the product digit, 4, to the next step.

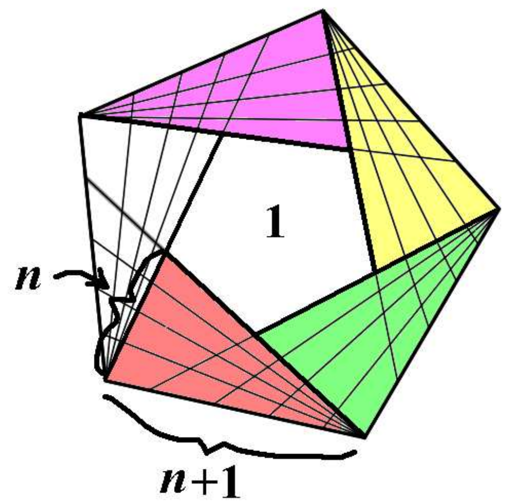
```
□001221  ← carries
□□102564 ← multiplicand
      ×4 ← multiplier
--------
□□410256 ← product
```

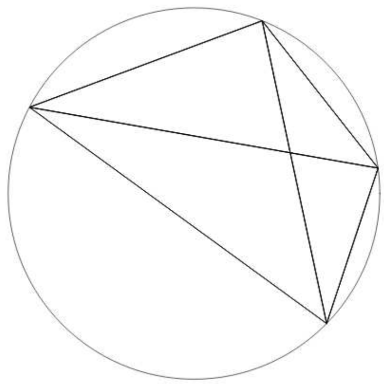
Harold could have written 102564. Note that Harold could have written 102564102564 or 102564102564102564, etc. But the question asks for the "smallest". The answer is 102564.



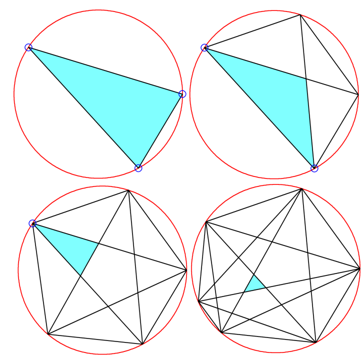
6. The spider′s divider On a regular pentagon, a spider forms segments that connect one endpoint of each side to n different non-vertex points on the side adjacent to the other endpoint of that side, going around clockwise, as shown. Into how many non-overlapping regions do the segments divide the pentagon? Your answer should be a formula involving n. (In the diagram, $n = 3$ and the pentagon is divided into 61 regions.)

Solution: Without any loss of generality, I can assume the spider divides each side into equal segments. This highlights the symmetry. In this pentagon, we can always find 5-way rotational symmetry, no matter what value of n is chosen. In the sketch at right, I have distinctly colored five congruent triangles which border the innermost pentagon. The number of regions in each triangle is as regular and countable as a rectangle divided by horizontal and vertical lines. (See sketch.) The final answer to this question is $n \times (n+1) \times 5 + 1 = 5n^2 + 5n + 1$.





9. Suppose n points on the circumference of a circle are joined by straight line segments in all possible ways and that no point that is not one of the original n points is contained in more than two of the segments. How many triangles are formed by the segments? Count all triangles whose sides lie along the segments, including triangles that overlap with other triangles. For example, for $n = 3$ there is one triangle and for $n = 4$ (shown in the diagram) there are 8 triangles.



Solution: Categorize the triangles by the number of vertices that lie on the circle. We have 4 cases.

3 vertices on the circle: One triangle is formed from any set of three points on the circle. If we have n points to choose from, this case contributes $\binom{n}{3}$ triangles.

2 vertices on the circle: The third vertex lies at a unique point inside the circle where two chords cross each other. This case requires four points on the circle. Among these four points, there are four pairs of adjacent points, each of which form a distinct triangle in a similar manner. If we have n points to choose from, this case contributes $4\binom{n}{4}$ triangles.

1 vertex on circle: Draw any triangle with one vertex on the circle and the other two vertices inside the circle. Then extend the sides of the triangle to intersect the circle. Some set of five points on the circle are required to create such a triangle. Then observe that this same set of five points creates a total of five triangles, depending which one of the five points is selected as a triangle vertex. If we have n points to choose from, this case contributes $5\binom{n}{5}$ triangles.

0 vertices on the circle: Draw any triangle with every vertex inside the circle. Then extend the sides of the triangle to intersect the circle. A set of six points on the circle is required to create such a triangle. Then observe that the only arrangement of these six points that form a triangle with no vertices on the circle has the points on the circle in the order A, B, C, A', B', C', where AA', BB', and CC' are the three line segments that are the extended sides of the triangle. If we have n points to choose from, this case contributes $\binom{n}{6}$ triangles.

The grand total number of triangles is $\binom{n}{3}+4\binom{n}{4}+5\binom{n}{5}+\binom{n}{6}$ triangles.

8. For what integer n does $x^2 - x + n$ divide into $x^{13} - 233x - 144$ with no remainder? That is, for what integer n is the first polynomial a factor of the second one? As always, justify your answer.

Solution 1: "Look for a pattern and try a simpler case"

Remark: Notice that 233 and 144 are members of the Fibonacci Sequence.

1	1	2	3	5	8	13	21	34	55	89	144	233
1*st*	2*nd*	3*rd*	4*th*	5*th*	6*th*	7*th*	8*th*	9*th*	10*th*	11*th*	12*th*	13*th*

Conjecture that $x^2 - x + n$ divides any $x^k - F_k x - F_{k-1}$, for odd k.

If the above conjecture is true, then

$x^2 - x + n$ divides $x^{13} - 233x - 144$,

$x^2 - x + n$ divides $x^{11} - 89x - 55$,

$x^2 - x + n$ divides $x^9 - 34x - 21$,

$x^2 - x + n$ divides $x^7 - 13x - 8$,

$x^2 - x + n$ divides $x^5 - 5x - 3$,

$x^2 - x + n$ divides $x^3 - 2x - 1$.

Let's add an additional conjecture that the same value of n satisfies all of the above. Factor $x^3 - 2x - 1$. We expect a quadratic factor times a linear factor. There are only two possible factorizations to consider.

$$x^3 - 2x - 1 = (x^2 - x - 1)(x + 1) \quad \text{or} \quad x^3 - 2x - 1 \neq (x^2 - x + 1)(x - 1)$$

If our conjectures are true, then $n = -1$. At this point, I would perform a polynomial long-division on the original 13th degree polynomial. It isn't nearly as tedious as one might expect. Especially if, while carrying out the long-division, you notice the coefficients of the quotients are forming a Fibonacci Sequence.

$$x^{13} - 233x - 144 = (x^2 - x - 1)(x^{11} + x^{10} + 2x^9 + 3x^8 + 5x^7 + 8x^6 + 13x^5 + 21x^4 + 34x^3 + 55x^2 + 89x + 144)$$

We haven't proven our conjectures, but we have demonstrated that $n = -1$ is a correct answer to the given problem.

Solution 2: If the first polynomial divides the second, then they will divide each other for every value of x. We choose various small values of x to see if we can narrow down the choices for n.

$x^2 - x + n \mid x^{13} - 233x - 144 \mid_{x=0} \Rightarrow n \mid -144$. $-144 = -2^4 \times 3^2$.

$n \in \{\pm 1, \pm 2, \pm 4, \pm 8, \pm 16, \pm 3, \pm 6, \pm 12, \pm 24, \pm 48, \pm 9, \pm 54, \pm 36, \pm 72, \pm 144\}$

$x^2 - x + n \mid x^{13} - 233x - 144 \mid_{x=1} \Rightarrow n \mid -376$. $-376 = -2^3 \times 47$.

$n \in \{\pm 1, \pm 2, \pm 4, \pm 8, \pm 47, \pm 94, \pm 188, \pm 376\}$

The previous two conditions must both be true, so $n \in \{\pm 1, \pm 2, \pm 4, \pm 8\}$. (Or, since n divides both 144 and 376, we could say n divides GCD(144, 376). $n|8$.)

$x^2 - x + n \mid x^{13} - 233x - 144 \mid_{x=-1} \Rightarrow n + 2 \mid +88$. $+88 = 2^3 \times 11$.

$n+2 \in \{\pm 1, \pm 2, \pm 4, \pm 8, \pm 11, \pm 22, \pm 44, \pm 88\}$
$n \in \{\ldots, -10, -6, -4, -3, -1, 0, +2, +6, +9, \ldots\}$

Since n is already constrained to lie between -8 and +8, I omitted the extreme choices. The intersection of the previous three tests yields $n \in \{-4, -1, +2\}$.

$$x^2 - x + n \mid x^{13} - 233x - 144 \,|_{x=2} \Rightarrow n+2 \mid +7582 . \quad +7582 = 2 \times 17 \times 223 .$$

$n+2 \in \{\pm 1, \pm 2 \pm 17, \pm 34, \pm 223, \pm 446, \pm 3791, \pm 7582\}$
$n \in \{\ldots, -4, -3, -1, 0, \ldots\}$

[Note that we don't actually have to complete the factorization of 7582. Due to the prior constraints on *n*, we only need to consider the very small factors of 7582.] Since *n* is already constrained to lie between -4 and +2, I omitted the extreme choices. The intersection of the previous four tests yields $n \in \{-4, -1\}$.

$$x^2 - x + n \mid x^{13} - 233x - 144 \,|_{x=-2} \Rightarrow n+6 \mid +7870 . \quad +7870 = 2 \times 5 \times 787 .$$

$n+6 \in \{\pm 1, \pm 2 \pm 5, \pm 10, \pm 787, \pm 1574, \pm 3935, \pm 7870\}$
$n \in \{\ldots, -16, -11, -8, -7, -5, -4, -1, +4, \ldots\}$

The intersection of the previous five tests remains $n \in \{-4, -1\}$.

At this point, I would perform polynomial long-divisions, starting with the smaller and easier choice of $n = -1$. As with Solution 1, the polynomial long-division may not be as tedious as one might expect.

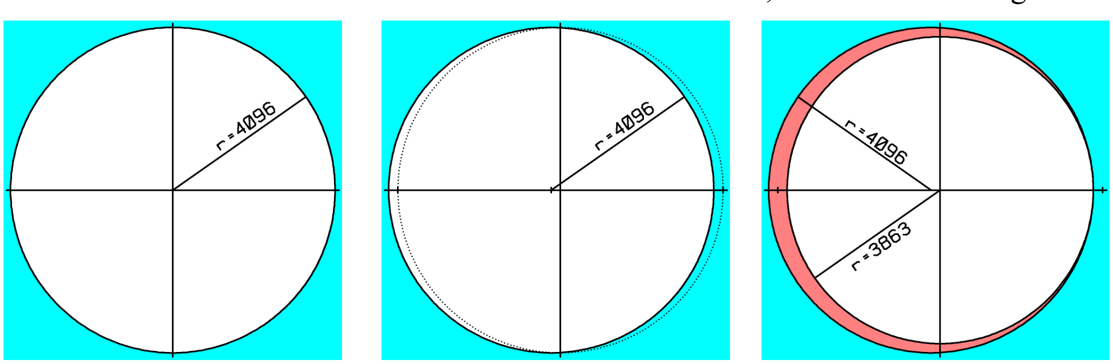
Solution 3: $x^{13} - 233x - 144 = 0$ has 13 roots. Descartes' Rule of Signs tell us there is exactly 1 positive real root, and there are either 2 or 0 negative real roots. (Zero is not a root, so all other roots are complex.) If you plot the polynomial at integer values $x \in \{-2, -1, 0, +1, +2\}$, you will find there are 2 negative real roots.

$x^2 - x + n = 0$ has 2 roots. By Descartes' Rule of Signs, if *n* is positive, we will have 2 or 0 positive real roots and 0 negative roots. (Since we can't have 2 positive real roots, both roots will be complex if *n* is positive.) And by Descartes' Rule of Signs, if *n* is negative we will have 1 positive real root and 1 negative real root. Although Descartes' Rule of Signs hasn't helped us much, it gives us a clue what to expect.

Rewriting the equation as $(x^{12} - 233)x = 144$, let's put a rough bound on what roots are plausible. If the magnitude of any complex root is ≥ 2, the magnitude of $x^{12} \geq 4096$. This means that x^{12} lies outside a circle, centered at the origin, of radius 4096 on the complex plane.

What is the value of $x^{12} - 233$? We take the circle of radius 4096 and shift it left by 233 units. All values of $x^{12} - 233$ are outside the shifted circle.

All values of $x^{12} - 233$ are also outside the smaller circle of radius 3863, centered at the origin.



We multiply by x again, and we find the magnitude of $(x^{12} - 233)x \geq 7726$. So, we can conclude that no root can have a magnitude of 2 or greater, because the left hand side of the equation, $(x^{12} - 233)x = 144$, would lie outside a circle, centered at the origin, of radius 7726 on the complex plane, while the real number 144 lies inside the circle.[1]

Next, let's examine the actual roots of $x^2 - x + n = 0$, constrained by $|x| < 2$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1-4n}}{2}.$$

Consider the discriminant, $D = b^2 - 4ac$, and which values of n correspond to real roots:

$$1 - 4n \geq 0 \Rightarrow 1 \geq 4n \Rightarrow n \leq 1/4.$$

What value of n corresponds to the largest possible real root?

$$\frac{1+\sqrt{1-4n}}{2} < +2 \Rightarrow \sqrt{1-4n} < 3 \Rightarrow 1-4n < 9 \Rightarrow -8 < 4n \Rightarrow -2 < n.$$

What value of n corresponds to the smallest possible real root?

$$-2 < \frac{1-\sqrt{1-4n}}{2} \Rightarrow -5 < -\sqrt{1-4n} \Rightarrow \sqrt{1-4n} < 5 \Rightarrow 1-4n < 25 \Rightarrow -24 < 4n \Rightarrow -6 < n.$$

All three of these conditions must be true if the quadratic is to have two real roots. Therefore, $-2 < n \leq 1/4$. We previously determined that zero was not a root of $x^{13} - 233x - 144 = 0$, so zero cannot be a root of $x^2 - x + n$. $n \neq 0$. If the quadratic has real roots, then $n = -1$. We might choose to test this solution, by division. Or we might move on to consider complex roots.

Restating the solution to our quadratic, $x = \dfrac{1 \pm \sqrt{1-4n}}{2}$

Consider the discriminant and which values of n correspond to complex roots:

$$1 - 4n < 0 \Rightarrow 1 < 4n \Rightarrow n > 1/4.$$

The constraint, $|x| < 2$, when applied to any of our complex roots means $\sqrt{\mathrm{Re}^2(x) + \mathrm{Im}^2(x)} < 2$. For our quadratic, $\mathrm{Re}(x) = \dfrac{1}{2}$, $\mathrm{Im}(x) = \dfrac{\sqrt{4n-1}}{2}$.

$\dfrac{1}{4} + \dfrac{4n-1}{4} < 4 \Rightarrow 1 + 4n - 1 < 16 \Rightarrow 4n < 16 \Rightarrow n < 4$. We are limited by the magnitude such that $1/4 < n < 4$. Or $n \in \{+1, +2, +3\}$.

Allowing for all possible real and complex roots, $n \in \{-1, +1, +2, +3\}$.

As with Solution 1, I would perform polynomial long-divisions, starting with the easier values of n.

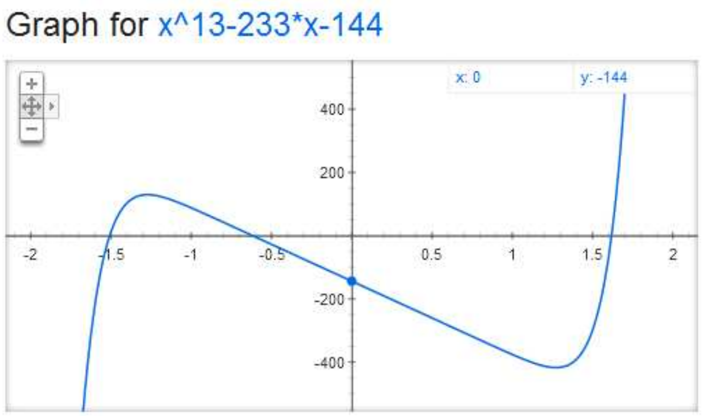
[1] Note that we could sharpen the bounds even more tightly. For example, consider |x|>5/3. Using big integer arithmetic or knowing 0.301<Log(2)<0.302, 0.477<Log(3)<0.478, you find |5/3|^12>400. Then, continuing with the same method shown above you can prove the equation is impossible. So, |x|<5/3.

A combination of the methods of Solution 2 with Solution 3 yields a unique solution, $n = -1$. If time permits, it would be wise to confirm your answer by long-division.

Note 1: The roots of the divisor polynomial, $x^2 - x - 1 = 0$, turn out to be, -phi and +Phi, where Phi (~1.618) and phi (~0.618) are the Golden Ratio and its reciprocal. The Golden Ratio and the Fibonacci Sequence often go hand-in-hand.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{+1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot a} = \frac{1}{2} \pm \frac{1}{2}\sqrt{5} \approx \{-0.618, +1.618\}$$

Note 2: The dividend polynomial is plotted on the *xy*-plane with the aid of Google.



Note 3: All 13 roots are plotted on the complex plane, with the aid of Mathematica. Our assertion in Solution 3 that all roots were within a circle of radius 2 is correct. In fact, one root is +Phi, and all other roots are strictly inside the circle of radius Phi.