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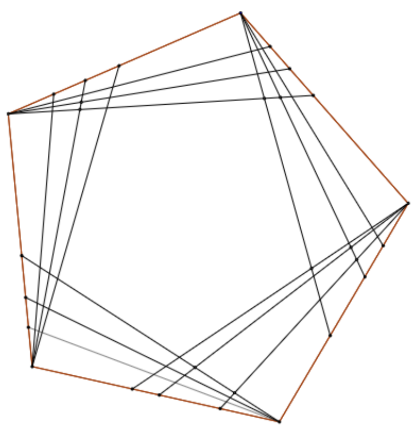
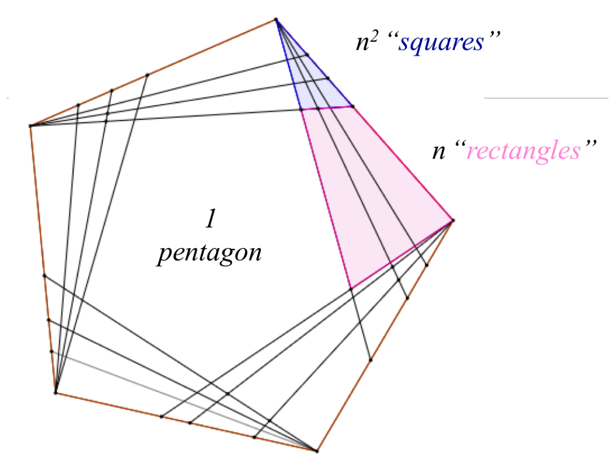
Picture Proofs

They say that success is a journey, not a destination. But I say, success is getting there as fast as you can, as gracefully as you can. Especially on a timed math contest. Today, I would like to share pictures proofs for two problems that math students have encountered this year.

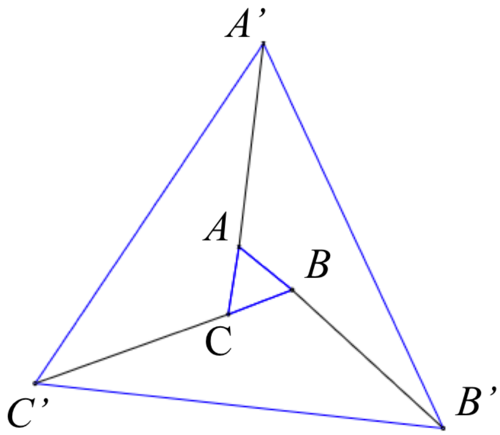
The first problem is called the spider's divider: on a regular pentagon, a spider forms segments that connect one endpoint of each side to n different non-vertex points on the side adjacent to the other endpoint of that side, going around clockwise, as shown. Into how many non-overlapping regions do the segments divide the pentagon? (2017 UNC Final Round Problem 6)



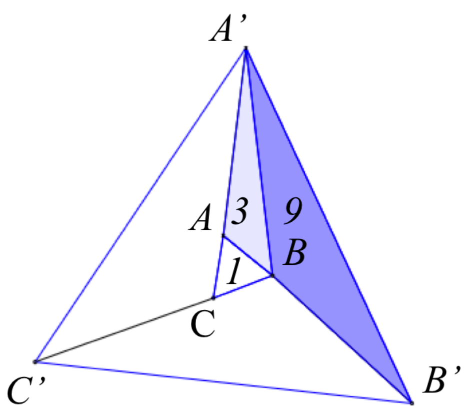
There are many ways that one can solve this, including recursion and summation techniques, algebra, and so on. But one elegant way is to count the shapes. For each triangle that lies on a side of the pentagon, there are n^2 square-like shapes and n rectangle-like shapes. Multiply $n^2 + n$ by 5 to account for all sides and add the 1 pentagon in the center, and you get the answer $5n^2 + 5n + 1$.



The second problem involves a triangle. Let ABC be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3AB$. Similarly, extend side $\overline{BC}$ beyond C to a point C' so that $CC' = 3BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3CA$. What is the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$? (2017 AMC10/12B)



Without loss of generality, let the area of ABC be 1. Since triangles ABA' and ABC share the same height, the ratio of their areas is equal to the ratio of their bases. So ABA' has area 3, which is three times that of ABC. Similarly, ABA' and $B'BA'$ share the same height, so $B'BA'$ has area 9. In total, $3(3+9)+1=37$ is the area of $A'B'C'$. $37:1$ is our answer.



Now, the expression $3(3+9)+1$ looks familiar. It looks similar to $5n^2 + 5n + 1$. In general, if we have our spider in the first problem divide a polygon with k sides, we would get $kn^2 + kn + 1$ non-overlapping regions. If we let the sides of the triangle in the second problem extend n times, we would get $3(n+n^2) + 1$, or $3n^2 + 3n + 1$, which is the same as letting $k = 3$ for the first problem.

All in all, picture proofs give us a simple way to solve problems that can give rise to surprising connections.