Twenty-sixth Annual UNC Math Contest **First Round Solutions** **Fall, 2017**

Rules: 90 minutes; no electronic devices.

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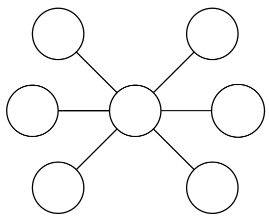
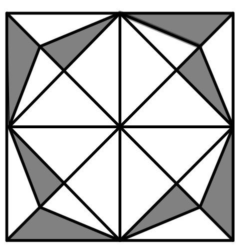
1. Find the largest integer n that satisfies both $61 < 5n$ and $n^2 < 199$.

Answer: $n = 14$
Solution: Since $5 \times 12 = 60$, we conclude from the first inequality that n must be greater than 12. Compute $13^2 = 169$, $14^2 = 196$, $15^2 = 225$ and deduce from the second inequality that n = 14.



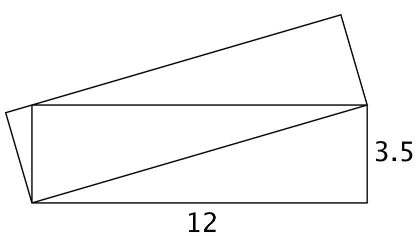
2. The seven integers 1, 2, 3, 5, 7, 9, and 11 are placed in the circles in the figure, one number in each circle and each number appearing exactly once. If all three straight-line sums are equal, then (a) what is that sum; and (b) what number is in the center circle?

Answer: (a) 14 (b) 2
Solution: The sum of all three straight-line sums is the sum of all the numbers plus two extra copies of the one in the middle. This sum must be three times the straight-line sum. The sum of all the numbers is 38. When we add two copies of any integer the result is an even number. We conclude that the straight-line sum must also be even. For a sum of any three different numbers from the list to be even, the three must include two odds and the 2. (There are other ways to see this by considering evenness and oddness.) Therefore, the middle number must 2. Conclude that the straight-line sum is $38 + 4 \div 3$ = $42 \div 3$=14. You can also fill the circles by trial and error.



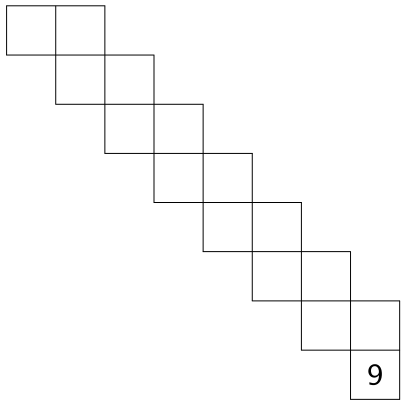
3. Find the total area of the eight shaded regions. The outer square has side length 10 and the octagon is regular, that is, its sides all have the same length and its angles are all congruent.

Answer: 25 square units
Solution: By symmetry, the unshaded square in the middle is one half of the total area of the big square. By symmetry, the shaded area is one half of the remaining area. The shaded area is one quarter of the area of the 10×10 square. The shaded region has area 100/4 =25.



4. The bottom rectangle in this figure is 3.5 units high and 12 units wide. How long is the shorter side of the upper (tilted) rectangle?

Answer: 84/25 units
Solution: The two rectangles have the same area and that area is 12×3.5=42. By the Pythagorean Theorem, the longer side of the tilted rectangle is $\sqrt{12^2 + (3.5)^2} = \sqrt{144 + 49/4} = \sqrt{625/4} = 25/2$. The shorter side is $42 \div (25/2)$= 84/25.



5. List the numbers from fifteen to one,
in permuted order, shrewdly done:
both across and down, in each line two neighbors share
every twosome sums to a perfect square.
If the last entry is the number 9,
find the leftmost entry in the top line.

Answer: 8
Solution: For each integer 1, 2, 3 ... 15, we can list the possible neighbors. For example, 1 can be next to 3 (to make 4), next to 8 (to make 9), or next to 15 (to make 16). Exactly two of the numbers, 8 and 9, have just one possible neighbor: 8 must be next to 1 and 9 must be next to 7. Therefore 8 and 9 must be the numbers at the ends. If 9 is at the bottom, then 8 must be at the top left.

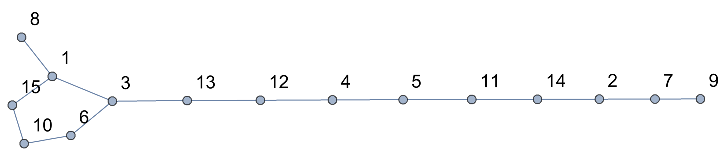


Figure 1: Diagram of Connected Pairs

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6. The Seripian unit of money is the pit, and Seripian coins come in only two types: 5-pit coins and 6-pit coins. What is the largest value that cannot be represented with Seripian coins? For example, 16 can be represented as $5 + 5 + 6$, but neither 8 nor 13 can be represented.

Answer: 19
Solution: Make a chart showing the possible combinations. First row is 0, 1, 2, 3, ... of the 5-pit coins and no 6-pit coins. Second row is 0, 1, 2, 3, ... of the 5 pit coins and one 6-pit coins, and so on, with one more 6-pit coin in each lower row.

0	5	10	15	20	25	30	35	...
6	11	16	21	26	31	36	41	...
12	17	22	27	32	37	42	47	...
18	23	28	33	38	43	48	53	...
24	29	34	39	44	49	54	59	...

The numbers that appear are, looking along diagonals, 5,6; 10,11,12; 15,16,17,18; 20,21,22,... The last integer that will be skipped is 19.

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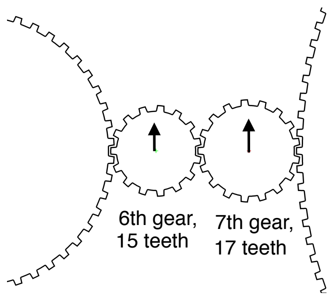
7. The table lists the number of teeth on each of thirteen consecutive intermeshed gears.

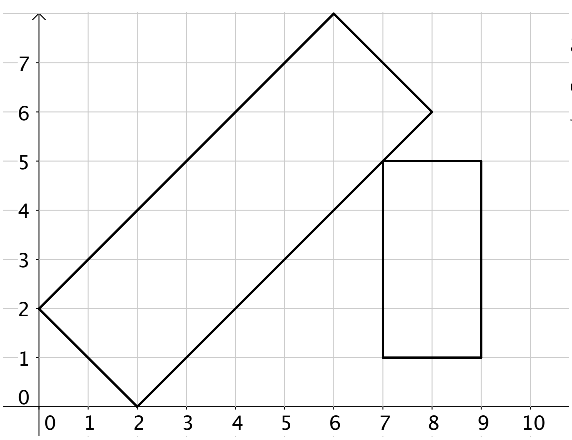
Gear :	1	2	3	4	5	6	7	8	9	10	11	12	13
Teeth :	144	36	24	60	48	15	17	144	72	34	12	12	144



Each gear is marked with an arrow, and initially all the arrows are pointing straight up. After how many revolutions of the first gear are all the arrows again pointing straight up for the first time? The diagram shows the sixth and seventh gears in the line.

Answer: 85

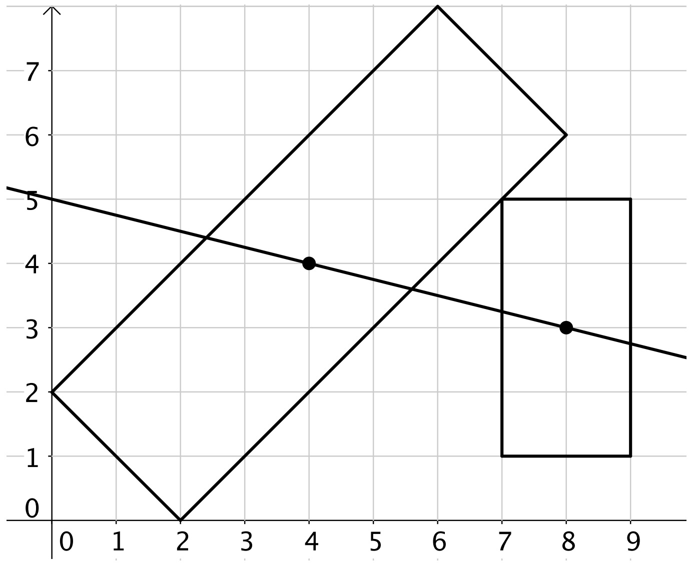
Solution: Call the number of revolutions of the first gear n. The arrow on a particular gear will point straight up when 144n is an integer multiple of the number of teeth on that gear. The answer will be the smallest integer n for which 144n is a multiple of each of the given integers. That is, we must choose n so that 144n is the least common multiple of all the given integers. Most of the integers are factors of 144; we can ignore those for the calculation. The integers that are not factors of 144 are 60, 15, 17, and 34. The least common multiple of 144, 60, 15, 17, and 34 is $5 \times 17 \times 144 = 85 \times 144$. Therefore n=85.



8. Draw one straight line that cuts both rectangles so that each of their individual areas is split in half. At what value of y does this line cross the y axis?

Answer: y=5

Solution: A line that goes through the center of a rectangle cuts the rectangle into two congruent pieces. The line that cuts the areas of both the rectangles in half is the line through the two centers.



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9. A point (x, y) whose coordinates x and y are both integers is called a *lattice point*. How many lattice points lie strictly inside the circle of radius π centered at the point $(0, 0)$? Recall that $\pi = 3.14159\ldots$

Answer: 29
Solution: It may be helpful to draw a sketch. The points (0,0), (0, $\pm$1), (0, $\pm$2), (0, $\pm$3), ($\pm$1, 0), ($\pm$2, 0), ($\pm$3, 0) are all clearly in the circle. So also are ($\pm$1, $\pm$1), ($\pm$1, $\pm$2), ($\pm$2, $\pm$1), and ($\pm$2, $\pm$2). The cases that need checking are ($\pm$1, $\pm$3) and ($\pm$3, $\pm$1). Use the Pythagorean theorem to see whether these lie inside or outside the circle- the distance from these points to (0,0) is $\sqrt{1+3^2} = \sqrt{10}$. If $\sqrt{10}$ is bigger than π then they are outside and if it is less than π then they are inside and should be added to the count. The number 3.15 is bigger than π and its square is 9.9225, which is less than 10. Therefore, $\sqrt{10}$ is bigger than π and the points are outside the circle. There are 29 lattice points inside the circle.

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10. Find a set of three consecutive odd integers $\{a, b, c\}$ for which the sum of squares $a^2 + b^2 + c^2$ is an integer made of four identical digits. (For example, 2222 is an integer made of four identical digits, and $\{7, 9, 11\}$ is a set of three consecutive odd integers.)

Answer: $41, 43, 45$
Solution: Let a=n, b=n+2, and c=n+4 and remember that n is odd. Then $a^2 + b^2 + c^2$= $3n^2 + 12n + 20$. This quantity is 1111, 2222, 3333, 4444, 5555, 6666, 7777, 8888, or 9999. Subtracting 20 from each of those, we find $3n^2 + 12n$ is either 1091, 2202, 3313, 4424, 5535, 6646, 7757, 8868, or 9979. Observe that $3n^2 + 12n$ is a multiple of 3 and eliminate the numbers on the list that are not multiples of three. This leaves only 2202, 5535, and 8868. Divide each of these by 3 and find that $n^2 + 4n$ is either 734, 1845, or 2956. Note that $n^2 + 4n$=n(n+4). This suggests looking at the factorizations of 734, 1845, and 2956. 734=2 $\times$ 367. 1845=5 $\times$ 9 $\times$ 41 = 41 $\times$ 45. 2956=2 $\times$ 2 $\times$ 739. Conclude that the three odd numbers we seek are 41, 43, 45.
(There are many ways to start. You can begin, for instance, by calling the three numbers 2n+1, 2n+3, and 2n+5 or by calling them n-2, n, and n+2 or by calling them 2n-1, 2n+1, and 2n+3. The reasoning will be similar whatever choice you make.)

NORTH EXIT	C1	C2	C3	C4	C5	C6
R1	.	.	.	.	29	30
R2						
R3	13	.	.	.	.	.
R4	12	11	10	9	8	7
R5	1	2	3	4	5	6

Attendant's Order of Packing Buses

11. A parking lot for 30 buses has 5 rows and 6 columns. Every day in January the buses depart heading north as columns: column C1 departs first, then C2 departs, . . ., so that the first bus to leave is the one in the first row, first column, the second bus out is the one in the second row, first column, etc. Each evening the buses return to the lot in their order of departure (first bus out is first bus in; etc.). An attendant parks the returning buses so they face north, filling the rows systematically, working from the southernmost row 5 to the northernmost row 1, in the snaking zig-zag order depicted. Call the locations of the buses on the morning of January 1 their original home positions. On what evening(s) in January will the attendant park the most buses in their original home positions?

Answer: The evenings of January 15 and January 30.

Solution: Using any system you like, number the buses and the parking spots and then check to see where each bus is parked each evening. It turns out to be less complicated than it may at first appear. You will find that 12 of the buses return to their home position every third evening, 8 of the buses return to their home position every fourth evening, and 10 of the buses return to their home position every fifth evening. This implies that 12+8 = 20 buses return to their home position every twelfth evening, 8+10 = 18 buses return to their home position every twentieth evening, 12+10 = 22 buses return to their home position every fifteenth evening, and all 30 buses return to their home position every sixtieth evening. Since there are only 31 days in January, there will not be an evening in January on which all 30 buses return to their home positions. The most buses that will be parked in their home positions on any single evening will be 22, and that will happen every fifteenth evening, namely, the evenings of January 15 and January 30. (See the solutions of Dr. Ming Song for one example of a numbering system.)

END OF CONTEST