Rules: Three hours; no electronic devices. Show work and justify answers.

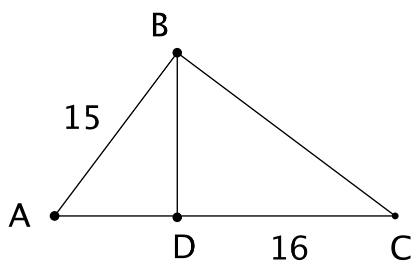
The positive integers are 1, 2, 3, 4, ...

---

1. A printer used 1890 digits to number all the pages in the Seripian Puzzle Book. How many pages are in the book? (For example, to number the pages in a book with twelve pages, the printer would use fifteen digits.)

ANSWER $\boxed{666 \text{ pages}}$

Solution Nine digits are used to number the first nine pages and $2 \times 90 = 180$ digits are used to number the ninety pages 10 to 99. This leaves $1890 - 189 = 1701$ digits. The next 900 pages will each use three digits, so $1701 \div 3 = 567$ more pages can be numbered. That is $9 + 90 + 567 = 666$ pages in all.



2. Segment AB is perpendicular to segment BC and segment AC is perpendicular to segment BD. If segment AB has length 15 and segment DC has length 16, then what is the area of triangle ABC?

ANSWER $\boxed{150}$

Solution Use similar triangles to deduce $\frac{AD}{15} = \frac{15}{AD+16}$. Clear denominators and obtain $AD^2 + 16AD = 15^2$. Solve to find $AD = 9$. This makes $AC = 25$ and triangle ABC is a 3-4-5 triangle. Deduce $BC = 20$ and conclude that the area is $(1/2)15 \times 20 = 150$ square units.

---

3. Find all values of B that have the property that if (x, y) lies on the hyperbola $2y^2 - x^2 = 1$, then so does the point $(3x + 4y, 2x + By)$.

ANSWER $\boxed{B = 3}$

Solution Suppose $2y^2 - x^2 = 1$ and find B so that

$1 = 2(2x + By)^2 - (3x + 4B)^2$

$= 8x^2 + 8Bxy + 2B^2y - 9x^2 - 24xy - 16y^2$

$= -x^2 + (8B - 24)xy + (2B^2 - 16)y^2$

Use the fact that $2y^2 - x^2 = 1$ to convert the line above to

$1 + (8B - 24)xy + (B^2 - 9)y^2$

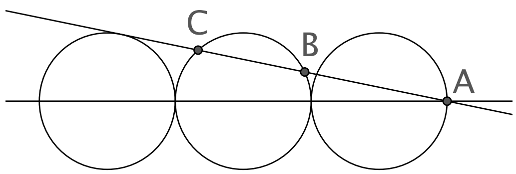
We want to find all values of B that make this equal to 1, or equivalently, so that $0 = (8B - 24)xy + (B^2 - 9)y^2$ for all (x, y) on the hyperbola. In particular, this must be true for $x = 0$ and $2y^2 = 1$, so $B^2 = 9$. This forces $8B = 24$ or $B = 3$.

---

4. How many positive integer factors of 36,000,000 are *not* perfect squares?

ANSWER $\boxed{149}$

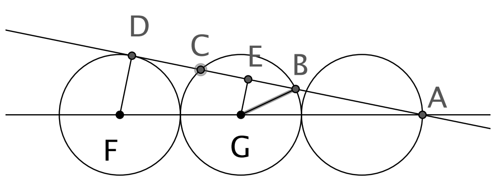
Solution All factors of $36,000,000 = 2^8 \cdot 3^2 \cdot 5^6$, have the form $2^a \cdot 3^b \cdot 5^c$ with $0 \le a \le 8, 0 \le b \le 2, 0 \le c \le 6$; and there are $9 \times 3 \times 7 = 189$ different choices. The perfect squares correspond to choosing even values for each of a, b, c; and there are $5 \times 2 \times 4 = 40$ such choices. Discarding the squares leaves $189 - 40 = 149$ factors that are not perfect squares.



5. Find the length of segment BC formed in the middle circle by a line that goes through point A and is tangent to the leftmost circle. The three circles in the figure all have radius one and their centers lie on the horizontal line. The leftmost and rightmost circles are tangent to the circle in the middle. Point A is at the rightmost intersection of the rightmost circle and the horizontal line.

ANSWER $\boxed{8/5}$

Solution



Draw the centers of the circle on the left and the one in the middle, here called F and G. Call the midpoint between B and C the point E. Then triangles FDA and GEA are similar. Conclude that FD/FA = GE/GA. It is given that FA=5 (count five radii), FD=1 and GA=3 (three radii). Therefore, GE= 3/5. Next look at the right triangle GEB. Hypotenuse GB has length 1. The leg GE has length 3/5. Use the pythagorean theorem (or observe GEB is a 3-4-5 triangle) to compute that EB= 4/5. Conclude CB has length 8/5.

---

6. **Circling the square.** Exactly one of these polynomials is a perfect square; that is, can be written as $(p(x))^2$ where $p(x)$ is also a polynomial. Circle the choice that is a perfect square, and for that choice, find the square root, the polynomial $p(x)$.

(A) $36 - 49x^2 + 14x^4$ (B) $36 - 48x^2 + 14x^4 - x^6$

(C) $9 - 12x + 4x^2 + 12x^3 - 8x^4 + 4x^6$ (D) $36 - 49x^2 + 15x^4 - x^6$

ANSWER $\boxed{C;\ p(x) = 2x^3 - 2x + 3}$

---

7. Let $x = 2^A + 10^B$ where A and B are randomly chosen with replacement from among the positive integers less than or equal to twelve. What is the probability that x is a multiple of 12?

ANSWER $\boxed{7/18}$

Solution Start by listing all the twelve values of the powers of 2 and 10, always reduced mod 12:

$m =$	1	2	3	4	5	6	7	8	9	10	11	12
$A = 2^m =$	2	4	8	4	8	4	8	4	8	4	8	4
$B = 10^m =$	10	4	4	4	4	4	4	4	4	4	4	4

The only pairs (A, B) for which $A + B$ is a multiple of 12 are $(A, B) = (2, 10)$ and $(A, B) = (8, 4)$. The first possibility has probability $\frac{1}{12} \times \frac{1}{12} = \frac{1}{144}$, and the second possibility has probability $\frac{5}{12} \times \frac{11}{12} = \frac{55}{144}$. Adding, we obtain a total probability of $\frac{1}{144} + \frac{55}{144} = \frac{56}{144} = \frac{7}{18}$

---

8. Let $p(x) = x^{2018} + x^{1776} - 3x^4 - 3$. Find the remainder when you divide $p(x)$ by $x^3 - x$.

ANSWER $\boxed{r(x) = -x^2 - 3}$

Solution Write $p(x) = q(x)(x^3 - x) + r(x)$ where the unknown remainder $r(x) = ax^2 + bx + c$ has three unknown coefficients.

The three roots of $x^3 - x$ are $x = -1, 0, 1$. Substituting these root values into the equations above, deduce that

$p(0) = -3 = c$;

$p(1) = -4 = a + b + c$; and

$p(-1) = -4 = a - b + c$.

The solution of this system is $(a, b, c) = (-1, 0, -3)$, so $r(x) = -x^2 - 3$.

---

9. Call a set of integers *Grassilian* if each of its elements is at least as large as the number of elements in the set. For example, the three-element set {2, 48, 100} is not Grassilian, but the six-element set {6, 10, 11, 20, 33, 39} is Grassilian. Let $\mathcal{G}(n)$ be the number of Grassilian subsets of $\{1, 2, 3, ..., n\}$. (By definition, the empty set is a subset of every set and is Grassilian.)

(a) Find $\mathcal{G}(3)$, $\mathcal{G}(4)$, and $\mathcal{G}(5)$.

(b) Find a recursion formula for $\mathcal{G}(n+1)$. That is, find a formula that expresses $\mathcal{G}(n+1)$ in terms of $\mathcal{G}(n), \mathcal{G}(n-1), \ldots$.

(c) Give an explanation that shows that the formula you give is correct.

ANSWER (a) $\boxed{\mathcal{G}(3) = 5, \;\; \mathcal{G}(4) = 8, \;\; \mathcal{G}(5) = 13}$

Solution (a) List the subsets and count the Grassilian ones.

ANSWER (b) $\boxed{\mathcal{G}(n) = \mathcal{G}(n-1) + \mathcal{G}(n-2); \;\; \mathcal{G}(0) = 1 \text{ and } \mathcal{G}(1) = 2}$

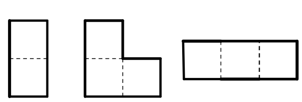
ANSWER (c) The sequence $5, 8, 13, \ldots$ in part (a) certainly suggests that Fibonacci numbers are afoot. Split the set of Grassilian subsets of $\{1, 2, 3, \ldots, n, n+1\}$ into two groups: those subsets that do contain the element $n + 1$ and those that do not. The ones that do not contain $n + 1$ are subsets of $\{1, 2, 3, \ldots, n\}$. In fact, that group is exactly the group of Grassilian subsets that is counted by $\mathcal{G}(n)$. Consider next the Grassilian subsets that do contain $n + 1$: there is a one-to-one correspondence between these and the Grassilian subsets of $\{1, 2, 3, \ldots, n-1\}$: None of these contain 1, for any set that contains both 1 and $n + 1$ has at least two elements and is not Grassilian. Associate to each of these sets the subset of

$\{1,2,3,\ldots,n-1\}$ whose elements are obtained by removing the element $n+1$ from the set and then subtracting 1 from each of the other elements. Note that the new sets are Grassilian subsets of $\{1,2,3,\ldots,n-1\}$. Each element is one less than an element of the original set and the number of elements is also one less. Therefore the number of such subsets is $\mathcal{G}(n-1)$. Conclude that $\mathcal{G}(n+1) = \mathcal{G}(n) + \mathcal{G}(n-1)$.
Alternate Solution.
To derive the recursion relation, consider a diagram that illustrates a typical case, say $n = 20$ and a subset S of size $k = 5$. To qualify as Grassilian, the five elements of S must be chosen from only among the acceptable choices $\{5,6,\ldots,20\}$.
The second line below shows the first five forbidden choices (marked X) and an example of an acceptable choice of five elements (marked x).

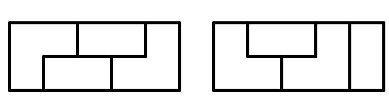
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
X	X	X	X	X		x			x				x	x				x		
						Xx			Xx				Xx	Xx				Xx		
					1	2	1	1	2	1	1	1	2	2	1	1	1	2	1	1

The third line transfers the X's to the right where they pair with xs. The fourth line encodes this pattern as a sequence of 1′ and 2s. This encoding shows that each Grassilian sets of size $k =$ five are in bijective correspondence with the ordered patterns of 1s and 2s whose sum is 21 and that contain exactly $k =$ five occurrences of 2s. The same can be said for the other possible choices of k.
Thus the number of Grassilian subsets of the first 20 integers is equal to $T(21)$, defined to be the number of ways that 21 can be expressed as an ordered sum of 1s and 2s. The number of such ordered sums can be shown to satisfy the recurrence relation $T(21) = T(20) + T(19)$ as follows:
every such ordered sum starts with either 1 or 2 (which are non-overlapping cases);
(i) if it starts with a "1", then the rest of the pattern must add to 20, and by definition there are $T(20)$ such patterns;
(ii) if it starts with a "2", then the rest of the pattern must add to 19, and by definition there are $T(19)$ such patterns.
****

This problem can also be approached by using binomials to count the subsets directly.
What we have called here Grassilian sets are sometimes called *Fat Sets*, a term introduced by George Andrews.

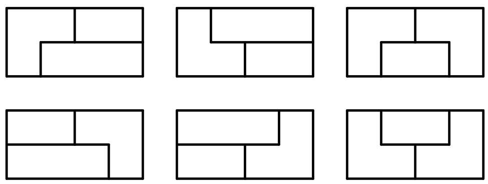


10. The Seripians have seen the error of their ways and issued new pit-coins in 2-pit and 3-pit denominations, containing 2 and 3 serigrams of gold. One of the new coins is in the shape of a domino (two adjoining squares) and the other two are in the shape of triominoes (three adjoining squares), shown above. To celebrate the new coins, the Seripians have announced a contest. Seripian students can win fame and glory and 100 of each of the new Seripian pit-coins by successfully completing quests (a)-(d) below.



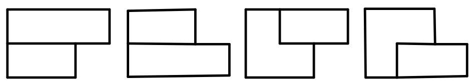
Call a tiling by pit-coins **prime** if there is no vertical line that splits the tiling into tilings of two smaller shapes without cutting across any of the coins. The

2x5 tiling above on the left is prime and the 2x5 tiling on the right is not prime.



Define $P(n)$ to be the number of distinct prime tilings of a horizontal 2xn grid. For example, $P(4) = 6$, and the six distinct prime 2x4 tilings are shown at left.

Define $Q(n)$ to be the number of distinct prime tilings of the two 2xn grids with one unit corner square missing at the right end. $Q(3) = 4$ and the four prime tilings are shown to the left. We wish you success on the **Seripian Quests**. Show your work.



(a) Determine $P(6)$.
(b) Determine formulas for $P(n)$ and $Q(n)$ in terms of $Q(n-1), Q(n-2)$, and/or $Q(n-3)$ that are valid for $n \geq 4$.
(c) Determine a formula for $P(n)$ that does not use Q. You may use $P(n-1), P(n-2)$, $P(n-3), \ldots$. Specify how large n must be for your formula to work.
(d) Determine explicitly $P(11)$ and $P(13)$.

ANSWER (a) $\boxed{P(6) = 10}$

Solution (a) List the 10 tilings or use your formula in part b.

ANSWER (b) $\boxed{P(n) = Q(n-1) + Q(n-2) \ \ (n \geq 4); Q(n) = Q(n-1) + Q(n-3) \ \ (n \geq 4)}$

Solution (b) Show your system for organizing and counting.
We first consider the Q. Divide the tilings of type Q for 2xn figures into two categories by considering what tile might form the overhang. The overhang will not be part of the L triomino, because a vertical line would separate the L from the rest of the tiling, so the tiling would not be prime. The overhang could be part of a domino or straight triple. The patterns for which the overhang is part of a domino each come from a unique prime pattern of Q type and one lower index. Moreover, a domino can be added in exactly one way to each prime tiling of the Q(n-1) type to form a prime tiling of the Q(n) type. Therefore, the number of tilings in which a domino forms the overhang is Q(n-1). If a prime tiling of the Q(n) type has its overhang formed by a triple, then the only possible structure that tiling can have has a second triple offset from that first. Each such tlings corresponds to exactly one tiling of the Q(n-3) type to which two offset triples have been appended. Thus, Q(n)=Q(n-1)+Q(n-3).
A similar reasoning applies to the P. Again divide the tilings of type P for 2xn figures into categories by considering what the right end could look like. For P, it is possible that the end is formed by an L coin. A P can be formed by appending an L in exactly one way to each tile of Q(n-1) type. There cannot be a vertical domino or the P is not prime. There could be a horizontal domino at the end. Above or below that, there cannot be another domino (or the tiling is not prime) or an L (or there is an untiled square and the tiling is not a P type). The tile above or below the domino has to be a triple. Therefore, the number of tilings of this type is equal to Q(n-2) and P(n)=Q(n-1)+Q(n-2).
The recursion for Q is known as the rule for Narayana's Cows, a sequence similar to the Fibonacci sequence. The sequence that obeys the recursion and begins 1, 1, 1, ... counts

Narayana's Cows. Start with one newborn cow and suppose that beginning in her fourth year, each cow produces a new female calf each year. The sequence counts the total number of cows each year: 1, 1, 1, 2 (the original calf is now in her fourth year and produces one female calf, so you have two cows), 3 (the original calf produces her second daughter, but last year's calf is not old enough to have her own yet), 4, 6 (now the calf born in the fourth year is old enough to have a calf), 9, 13, 19, ...

ANSWER (c) $\boxed{P(n) = P(n-1) + P(n-3) \ \ (n \geq 7)}$

Solution (c) One way to see this is to note that since the Q satisfy the Narayana recursion and P is a sum of two Q, then the P also satisfies the recursion. One can also verify this by writing P in terms of Q and checking.

ANSWER (d) $\boxed{P(11) = 74; P(13) = 158}$

Solution (d) Use either (b) or (c) to compute. Take care about the low values. The first ones must be computed directly.

$n =$	1	2	3	4	5	6	7	8	9	10	11	12	13
$Q =$	0	2	4	4	6	10	14	20	30	44	64	108	172
$P =$	1	1	3	6	8	10	16	24	34	50	74	108	158

---

11. (a) Find an integer $n > 1$ for which $1 + 2 + \ldots + n^2$ is a perfect square.

(b) Show that there are infinitely many integers $n > 1$ that have the property that $1 + 2 + \ldots + n^2$ is a perfect square, and determine at least three more examples of such n. Hint: There is one approach that uses the result of a previous problem on this contest.

ANSWER (a) $\boxed{7}$ (Other acceptable answers are 41, 239, 1393, 8119, and, in general, anything generated by the formula in part b. The answer students are most likely to give is 7.)

Solution (a) $1 + 2 + \ldots + n^2 = \frac{n^2(n^2+1)}{2}$. This will be the square of an integer exactly if $\frac{n^2+1}{2}$ is the square of an integer. We seek integers n and k that satisfy $n^2 + 1 = 2k^2$. Because the right hand side is even, all solutions n are odd. Examine small odd integers n=3, 5, ... and conclude that $n = 7$ works. For $n = 7$, $n^2 + 1 = 50$, so $k = 5$. The next integer n that works is $n = 41$. See part (b) for more values of n.

ANSWER/*Solution* (b) From question 3 above (about the hyperbola), we know that if (n,k) satisfies $n^2 + 1 = 2k^2$, then so also does $(3n + 4k, 2n + 3k)$. Moreover, if n and k are integers, then so also are 3n+4k and 2n+3k. Therefore, iterating the formula produces an infinite number of integer solutions. This is what we are asked to prove.

We can compute the first few. Begin with n=7, k=5:

$(3n + 4k, 2n + 3k)$= (21+20, 14+15) = (41, 29).

Taking n=41 and k=29,

$(3n + 4k, 2n + 3k)$= (123+116, 82+87)=(239,169)

Now take n=239 and k=169.

$(3n + 4k, 2n + 3k)$=(717+676, 498+507)=(1393, 985)

Therefore 41, 239, 1393 are three more possible values for n.

This can also be approached with continued fractions. We seek integers that solve $n^2 + 1 = 2k^2$. Consider large values of n and k and recast the equation as $(n/k)^2 + (1/k^2) = 2$: we are seeking rational numbers n/k that approximate, and slightly underestimate, $\sqrt{2}$. The continued fraction for $\sqrt{2}$ provides such rationals.
The equation $n^2 + 1 = 2k^2$, to be solved in integers, is an example of a negative Pell equation.

END OF CONTEST