Rules: Three hours; no electronic devices. Show work and justify answers.

The positive integers are 1, 2, 3, 4, . . .

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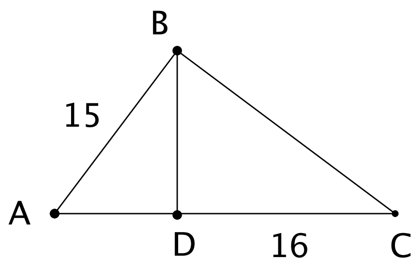
1. A printer used 1890 digits to number all the pages in the Seripian Puzzle Book. How many pages are in the book? (For example, to number the pages in a book with twelve pages, the printer would use fifteen digits.)

ANSWER 666 pages



2. Segment AB is perpendicular to segment BC and segment AC is perpendicular to segment BD. If segment AB has length 15 and segment DC has length 16, then what is the area of triangle ABC?

ANSWER 150

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3. Find all values of B that have the property that if (x, y) lies on the hyperbola $2y^2 - x^2 = 1$, then so does the point $(3x + 4y, 2x + By)$.

ANSWER $B = 3$

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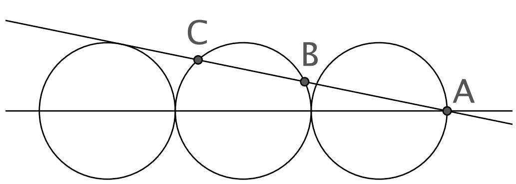
4. How many positive integer factors of $36,000,000$ are *not* perfect squares?

ANSWER 149



5. Find the length of segment BC formed in the middle circle by a line that goes through point A and is tangent to the leftmost circle. The three circles in the figure all have radius one and their centers lie on the horizontal line. The leftmost and rightmost circles are tangent to the circle in the middle. Point A is at the rightmost intersection of the rightmost circle and the horizontal line.

ANSWER 8/5

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6. **Circling the square.** Exactly one of these polynomials is a perfect square; that is, can be written as $(p(x))^2$ where $p(x)$ is also a polynomial. Circle the choice that is a perfect square, and for that choice, find the square root, the polynomial $p(x)$.

(A) $36 - 49x^2 + 14x^4$ (B) $36 - 48x^2 + 14x^4 - x^6$

(C) $9 - 12x + 4x^2 + 12x^3 - 8x^4 + 4x^6$ (D) $36 - 49x^2 + 15x^4 - x^6$

ANSWER $\boxed{\text{C; } p(x) = 2x^3 - 2x + 3}$

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7. Let $x = 2^A + 10^B$ where A and B are randomly chosen with replacement from among the positive integers less than or equal to twelve. What is the probability that x is a multiple of 12?

ANSWER $\boxed{7/18}$

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8. Let $p(x) = x^{2018} + x^{1776} - 3x^4 - 3$. Find the remainder when you divide $p(x)$ by $x^3 - x$.

ANSWER $\boxed{r(x) = -x^2 - 3}$

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9. Call a set of integers *Grassilian* if each of its elements is at least as large as the number of elements in the set. For example, the three-element set {2, 48, 100} is not Grassilian, but the six-element set {6, 10, 11, 20, 33, 39} is Grassilian. Let $\mathcal{G}(n)$ be the number of Grassilian subsets of $\{1, 2, 3, ..., n\}$. (By definition, the empty set is a subset of every set and is Grassilian.)

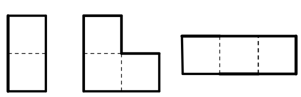
(a) Find $\mathcal{G}(3)$, $\mathcal{G}(4)$, and $\mathcal{G}(5)$.

(b) Find a recursion formula for $\mathcal{G}(n+1)$. That is, find a formula that expresses $\mathcal{G}(n+1)$ in terms of $\mathcal{G}(n), \mathcal{G}(n-1), \ldots$.

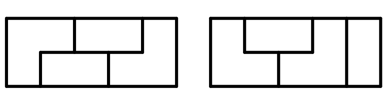
(c) Give an explanation that shows that the formula you give is correct.

ANSWER (a) $\boxed{\mathcal{G}(3) = 5, \ \ \mathcal{G}(4) = 8, \ \ \mathcal{G}(5) = 13}$

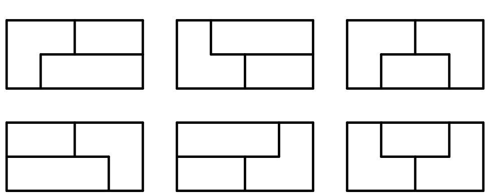
ANSWER (b) $\boxed{\mathcal{G}(n) = \mathcal{G}(n-1) + \mathcal{G}(n-2); \ \ \mathcal{G}(0) = 1 \text{ and } \mathcal{G}(1) = 2}$



10. The Seripians have seen the error of their ways and issued new pit-coins in 2-pit and 3-pit denominations, containing 2 and 3 serigrams of gold. One of the new coins is in the shape of a domino (two adjoining squares) and the other two are in the shape of triominoes (three adjoining squares), shown above. To celebrate the new coins, the Seripians have announced a contest. Seripian students can win fame and glory and 100 of each of the new Seripian pit-coins by successfully completing quests (a)-(d) below.

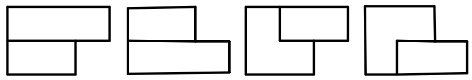


Call a tiling by pit-coins **prime** if there is no vertical line that splits the tiling into tilings of two smaller shapes without cutting across any of the coins. The 2x5 tiling above on the left is prime and the 2x5 tiling on the right is not prime.



Define $P(n)$ to be the number of distinct prime tilings of a horizontal 2xn grid. For example, $P(4) = 6$, and the six distinct prime 2x4 tilings are shown at left.

Define $Q(n)$ to be the number of distinct prime tilings of the two 2xn grids with one unit



corner square missing at the right end. $Q(3) = 4$ and the four prime tilings are shown to the left. We wish you success on the **Seripian Quests**. Show your work.

(a) Determine $P(6)$.
(b) Determine formulas for $P(n)$ and $Q(n)$ in terms of $Q(n-1), Q(n-2)$, and/or $Q(n-3)$ that are valid for $n \geq 4$.
(c) Determine a formula for $P(n)$ that does not use Q. You may use $P(n-1), P(n-2)$, $P(n-3), \ldots$. Specify how large n must be for your formula to work.
(d) Determine explicitly $P(11)$ and $P(13)$.

ANSWER (a) $\boxed{P(6) = 10}$

ANSWER (b) $\boxed{P(n) = Q(n-1) + Q(n-2) \ \ (n \geq 4); Q(n) = Q(n-1) + Q(n-3) \ \ (n \geq 4)}$

ANSWER (c) $\boxed{P(n) = P(n-1) + P(n-3) \ \ (n \geq 7)}$

ANSWER (d) $\boxed{P(11) = 74; P(13) = 158}$

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11. (a) Find an integer $n > 1$ for which $1 + 2 + \ldots + n^2$ is a perfect square.
(b) Show that there are infinitely many integers $n > 1$ that have the property that $1 + 2 + \ldots + n^2$ is a perfect square, and determine at least three more examples of such n. Hint: There is one approach that uses the result of a previous problem on this contest.

ANSWER (a) $\boxed{7}$ (Other acceptable answers are 41, 239, 1393, 8119, and, in general, anything generated by the formula in part b. The answer we expect to see students give is 7.)
ANSWER (b) 41, 239, 1393 are three more possible values for n.

END OF CONTEST