University of Northern Colorado Mathematics Contest 2017-2018

Problems and Solutions of Final Round

1. A printer used 1890 digits to number all the pages in the Seripian Puzzle Book. How many pages are in the book? (For example, to number the pages in a book with twelve pages, the printer would use fifteen digits.)

Answer: 666

Solution:

Let *n* be the number of pages of the book.

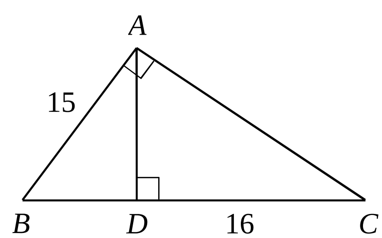
There are 9 one-digit numbers and 90 two-digit numbers. $9 + 2 \cdot 90 = 189$ digits are used for these 99 pages.

$1890 - 189 = 1701$ digits are used to number the pages from page 100 to page *n*.

We have $(n - 99) \cdot 3 = 1701$.

We obtain $n = 666$.

2. Segment *AB* is perpendicular to segment *BC* and segment *AC* is perpendicular to segment *BD*. If segment *AB* has length 15 and segment *DC* has length 16, then what is the area of triangle *ABC*?



Answer: 150

Solution:

If we recognize that all triangles are 3-4-5 triangles. Immediately we have $B \quad = 9$, $A \quad = 12$, and $B \quad = 25$.

The area of triangle *ABC* is $\frac{1}{2} \cdot 25 \cdot 12 = 150$.

Let us solve for *BD*.

Let $B \quad = x$.

Triangles *ADB* and *CAB* are similar.

We have $\frac{B}{B} = \frac{B}{B}$. That is, $\frac{x}{1} = \frac{1}{x+1}$.

Solving for *x* we obtain $x = 9$.

3. Find all values of B that have the property that if (x, y) lies on the hyperbola

$$2y^2 - x^2 = 1,$$

then so does the point $(3x + 4y,\ 2x + B\ )$.

Answer: one value: 3

Solution 1:

We know that $(1, 1)$ is on the hyperbola. Then $(7, 2 + B)$ is.

So $2(2 + B)^2 - 7^2 = 1$. Then $(2 + B)^2 = 25$. We obtain $B = 3$ or $B = -7$.

If we plug $(3x + 4y,\ 2x + 3y)$ into the original equation, we will obtain the original equation.

So $B = 3$ is valid.

Now we plug $(3x + 4y,\ 2x - 7y)$ into the original equation. We don't get the original equation.

So $B = -7$ is invalid.

There is only one value for *B*.

Solution 2:

Plug $(3x + 4y,\ 2x + B\ )$ into the original equation:

$$2(2x + B\ )^2 - (3x + 4y)^2 = 1$$

Expand the left side and collect the like terms:

$$(2B^2 - 16)y^2 - x^2 + (8B - 24)x\ \ = 1$$

In the above equation the cross term *xy* must disappear. So $B = 3$.

Then the above equation becomes the original hyperbola.

There is only one value for *B*.

4. How many positive integer factors of 36,000,000 are not perfect squares?

Answer: 149

Solution:

The prime factorization of 36,000,000 is

$$36{,}000{,}000 = 2^8 \cdot 3^2 \cdot 5^6$$

It has $(8 + 1) \cdot (2 + 1) \cdot (6 + 1) = 189$ factors where $a \le 8, b \le 2, c \le 6$.

Let $2^a \cdot 3^b \cdot 5^c$ be a factor of 36,000,000.

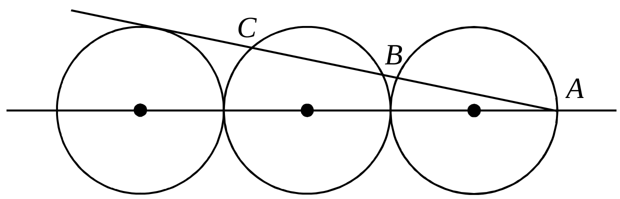
For it to be a square, we need all of *a*, *b*, and *c* to be even.

There are five choices for *a*: 0, 2, 4, 6, 8; two choices for *b*: 0, 2; and four choices for *c*: 0, 2, 4, 6 such that the factor is a square.

So 36,000,000 has $5 \cdot 2 \cdot 4 = 40$ factors that are squares.

The number of the non-square factors of 36,000,000 is $189 - 40 = 149$.

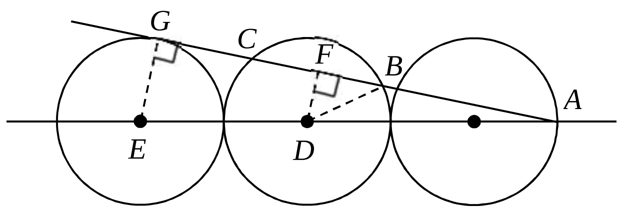
5. Find the length of segment *BC* formed in the middle circle by a line that goes through point *A* and is tangent to the leftmost circle. The three circles in the figure all have radius one and their centers lie on the horizontal line. The leftmost and rightmost circles are tangent to the circle in the middle. Point *A* is at the rightmost intersection of the rightmost circle and the horizontal line.



Answer: $\frac{8}{5}$

Solution:

Let *D* be the center of the middle circle, and *E* be the center of the leftmost circle.



Draw *BD*.

Draw $D \quad \perp B$ with *F* on line *ABC*.

Draw $E \quad \perp B$ with *G* on line *ABC*. G is the tangent point.

Triangles *AFD* and *AGE* are similar with $E \quad = 1$, $A \quad = 3$, and $A \quad = 5$.

So $D \quad = \frac{3}{5}$.

With $B \quad = 1$ triangle *BFD* is a 3-4-5 triangle.

So $B \quad = \frac{4}{5}$.

Then $B \quad = \frac{8}{5}$.

6. **Circling the square.** Exactly one of these polynomials is a perfect square; that is, can be written as $(p(x))^2$ where $p(x)$ is also a polynomial. Circle the choice that is a perfect square, and for that choice, find the square root, the polynomial $p(x)$.

(A) $36 - 49x^2 + 14x^4$

(B) $36 - 48x^2 + 14x^4 - x^6$

(C) $9 - 12x + 4x^2 + 12x^3 - 8x^4 + 4x^6$

(D) $36 - 49x^2 + 15x^4 - x^6$

Answer: (C) with $p(x) = 2x^3 - 2x + 3$

Solution:

(A) $36 - 49x^2 + 14x^4$

Obviously, it is not the square of a polynomial.

(B) $36 - 48x^2 + 14x^4 - x^6$

The leading coefficient of the square of a polynomial must be positive.

So it is not a square.

(C) $9 - 12x + 4x^2 + 12x^3 - 8x^4 + 4x^6$

Let

$$9 - 12x + 4x^2 + 12x^3 - 8x^4 + 4x^6 = (2x^3 + ax^2 + b \quad + 3)^2$$

Since there is no term of x^5, $a = 0$.

Then

$$9 - 12x + 4x^2 + 12x^3 - 8x^4 + 4x^6 = (2x^3 + b \quad + 3)^2.$$

Expanding the right side we will get $b = -2$.

We have

$$9 - 12x + 4x^2 + 12x^3 - 8x^4 + 4x^6 = (2x^3 - 2x + 3)^2.$$

So

$$p(x) = 2x^3 - 2x + 3$$

(D) $36 - 49x^2 + 15x^4 - x^6$

The leading coefficient is negative.

It is not a square.

7. Define $x = 2^A + 10^B$ where A and B are randomly chosen with replacement from among the positive integers less than or equal to twelve. What is the probability that x is a multiple of 12?

Answer: $\frac{7}{1}$

Solution:

For x to be divisible by 4, we must have $A = 1$ and $B = 1$ or $A \geq 2$ and $B \geq 2$.

Case 1: $A = 1$ and $B = 1$

The probability is $\frac{1}{1} \cdot \frac{1}{1}$.

Now $x = 12$ is divisible by 12.

The probability in this case is

$$\frac{1}{12} \cdot \frac{1}{12} = \frac{1}{144}$$

Case 2: $A \geq 2$ and $B \geq 2$

The probability is $\frac{1}{1} \cdot \frac{1}{1}$.

Since 10^B is 1 mod 3, we need 2^A to be 2 mod 3.

So A must be odd.

There are 5 values out of 11 for A: 3, 5, 7, 9, 11.

The probability in this case is

$$\frac{11}{12}\cdot\frac{11}{12}\cdot\frac{5}{11}=\frac{55}{144}$$

The total probability is

$$\frac{1}{144}+\frac{55}{144}=\frac{56}{144}=\frac{7}{18}$$

8. Let $p(x) = x^2 \quad + x^1 \quad - 3x^4 - 3$. Find the remainder when you divide $p(x)$ by $x^3 - x$.

Answer: $-x^2 - 3$

Solution 1:

Note that $x^n = x^{n-2} \bmod x^3 - x$ for all $n \geq 3$:

$$x^n - x^{n-2} = x^{n-3}(x^3 - x) = 0 \bmod x^3 - x.$$

By induction, $x^n = x^{n-2m} \bmod x^3 - x$ for all $n \geq 3$ if $n - 2m \geq 1$.

So

$$\begin{aligned} p(x) = x^2 \quad + x^1 \quad - 3x^4 - 3 &= x^1 \quad + x^1 \quad - 3x^4 - 3 = 2x^1 \quad - 3x^4 - 3 \\ &= 2x^4 - 3x^4 - 3 = -x^4 - 3 = -x^2 - 3 \bmod x^3 - x \end{aligned}$$

The answer is $-x^2 - 3$.

Solution 2:

Note that $x^3 - x$ is a polynomial of degree 3.

We let $ax^2 + b \quad + c$ be the remainder.

Then

$$x^2 \quad + x^1 \quad - 3x^4 - 3 = (x^3 - x)Q(x) + ax^2 + b \quad + c$$

for some polynomial $Q(x)$.

Let $x = 0$:

$$-3 = c$$

Let $x = 1$:

$$-4 = a + b + c$$

Let $x = -1$:

$$-4 = a - b + c$$

Solving for a, b, and c we obtain $a = -1$, $b = 0$, and $c = -3$.

The remainder is $-x^2 - 3$.

9. Call a set of integers *Grassilian* if each of its elements is at least as large as the number of elements in the set. For example, the three-element set $\{2, 48, 100\}$ is not Grassilian, but the six-element set $\{6, 10, 11, 20, 33, 39\}$ is Grassilian. Let $G(n)$ be the number of Grassilian subsets of $\{1, 2, 3, \cdots, n\}$. (By definition, the empty set is a subset of every set and is Grassilian.)

   (a) Find $G(3)$, $G(4)$, and $G(5)$.

   (b) Find a recursion formula for $G(n+1)$. That is, find a formula that expresses $G(n+1)$ in terms of $G(n)$, $G(n-1)$, $\cdots$.

   (c) Give an explanation that shows that the formula you give is correct.

Answer:

(a) $G(5) = 3$, $G(4) = 8$, $G(5) = 13$

(b) $G(n+1) = G(n) + G(n-1)$

Solution:

$\{1\}$ has 2 Grassilian subsets: $\{\ \}$, $\{1\}$.

$\{1, 2\}$ has 3 Grassilian subsets: $\{\ \}$, $\{1\}$, $\{2\}$.

$\{1, 2, 3\}$ has 5 Grassilian subsets: $\{\ \}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{2, 3\}$

…

By simply listing we find

$$G(1) = 2, G(2) = 3, G(3) = 5, G(4) = 8 \text{ and } G(5) = 13.$$

We conjecture that it is the Fibonacci sequence:

$$G(1) = 2,\ G(2) = 3, \text{ and } G(n+1) = G(n) + G(n-1) \text{ for } n \geq 2$$

Let us prove that the Fibonacci sequence is the correct pattern for this problem by building the recursion.

$G(n+1)$ is the number of Grassilian subsets of $\{1, 2, 3, \cdots, n-1, n, n+1\}$.

There are two cases for the subsets.

Case 1: $n+1$ is not in the subset

Any Grassilian subset of $\{1, 2, 3, \cdots, n-1, n\}$ is a Grassilian subset of $\{1, 2, 3, \cdots, n-1, n, n+1\}$.

There are $G(n)$ Grassilian subsets in this case.

Case 2: $n+1$ is in the subset

For any Grassilian subset $\{a_1, a_2, a_3, \cdots, a_k\}$ of $\{1, 2, 3, \cdots, n-1\}$, we have

$\{a_1 + 1, a_2 + 1, a_3 + 1, \cdots, a_k + 1, n+1\}$ as a Grassilian subset of $\{1, 2, 3, \cdots, n-1, n, n+1\}$.

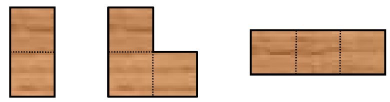
For example, $\{\ \}$, $\{1\}$, $\{2,4\}$, and $\{3, 4, 5\}$ are Grassilian subsets of $\{1, 2, 3, 4, 5\}$. Then $\{7\}$, $\{2,7\}$, $\{3, 5, 7\}$, and $\{4, 5, 6, 7\}$ are Grassilian subsets of $\{1, 2, 3, 4, 5, 6, 7\}$.

There are $G(n-1)$ Grassilian subsets in this case.

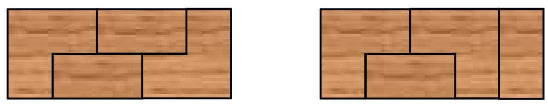
Therefore,

$$G(n+1) = G(n) + G(n-1).$$

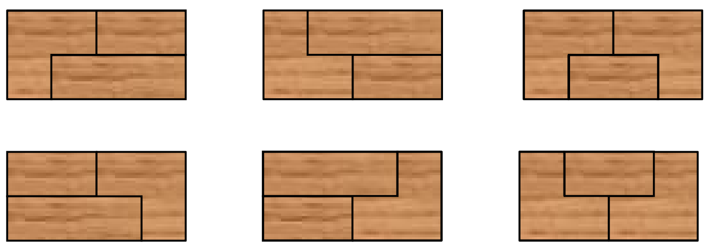
10. The Seripians have seen the error of their ways and issued new pit-coins in 2-pit and 3-pit denominations, containing 2 and 3 serigrams of gold. One of the new coins is in the shape of a domino (two adjoining squares) and the other two are in the shape of triominoes (three adjoining squares), shown above. To celebrate the new coins, the Seripians have announced a contest. Seripian students can win fame and glory and 100 of each of the new Seripian pit-coins by successfully completing quests (a)-(d) below.



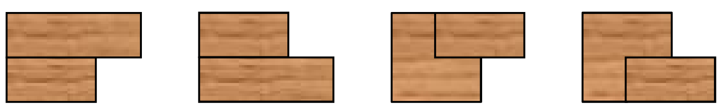
Call a tiling by pit-coins *prime* if there is no vertical line that splits the tiling into tilings of two smaller shapes without cutting across any of the coins. The 2×5 tiling below on the left is prime and the 2×5 tiling on the right is not prime.



Define $P(n)$ to be the number of distinct prime tilings of a horizontal $2 \times n$ grid. For example, $P(4) = 6$, and the six distinct prime 2×4 tilings are shown below.



Define $Q(n)$ to be the number of distinct prime tilings of the two $2 \times n$ grids with one unit corner square missing at the right end. $Q(3) = 4$ and the four prime tilings are shown below.



We wish you success on the Seripian Quests. Show your work.

(a) Determine $P(6)$.

(b) Determine formulas for $P(n)$ and $Q(n)$ in terms of $Q(n-1)$, $Q(n-2)$, and/or $Q(n-3)$ that are valid for $n \geq 4$.

(c) Determine a formula for $P(n)$ that does not use Q. You may use $P(n-1)$, $P(n-2)$, $P(n-3)$, $\cdots$.

Specify how large *n* must be for your formula to work.

(d) Determine explicitly $P(11)$ and $P(13)$.

Answer:

(a) $P(6) = 10$

(b) $P(n) = Q(n-1) + Q(n-2)$, $Q(n) = Q(n-1) + Q(n-3)$

(c) $P(n) = P(n-1) + P(n-3)$ for $n \geq 7$

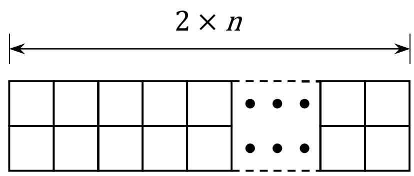
(d) $P(11) = 74$, $P(13) = 158$

Solution:

Note that $Q(n)$ is the number of distinct prime tilings of the two $2 \times n$ grids with one unit corner square missing at the right end.

So $\frac{Q(n)}{2}$ is the number of distinct prime tilings of one $2 \times n$ grid with any particular corner square missing.

This is a $2 \times n$ grid.

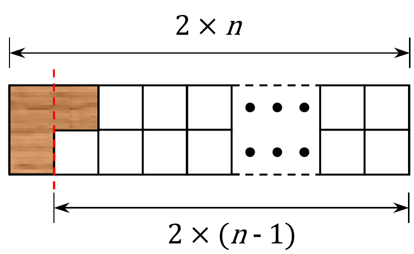


To tile it there are $P(n)$ ways.

Let us build the recursion for $P(n)$.

There are four cases to tile the top-left square.

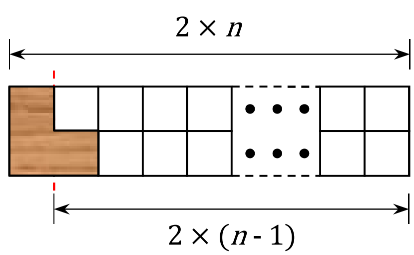
Case 1:



After we use the first coin, we have $2 \times (n-1)$ grid with a missing corner square left to be tiled.

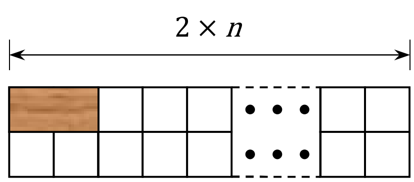
There are $\frac{Q(n-1)}{2}$ ways to tile the rest.

Case 2:

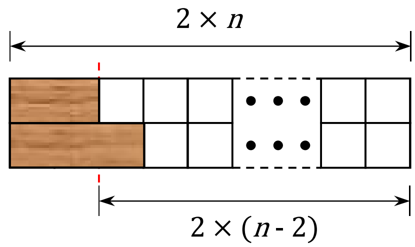


There are $\frac{Q(n-1)}{2}$ ways to tile the rest.

Case 3:

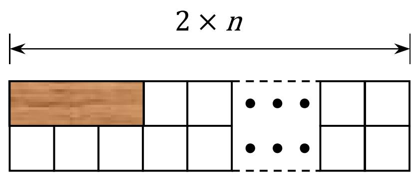


In this case there is only one way to tile the bottom-left square:

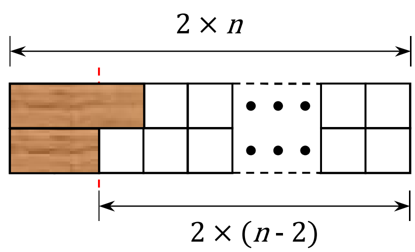


There are $\frac{Q(n-2)}{2}$ ways to tile the rest.

Case 4:



In this case there is only one way to tile the bottom-left square:



There are $\frac{Q(n-2)}{2}$ ways to tile the rest.

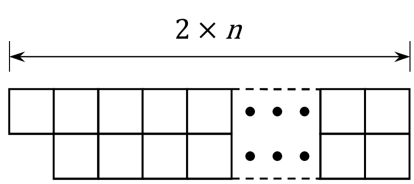
Therefore,

$$P(n) = \frac{Q(n-1)}{2} + \frac{Q(n-1)}{2} + \frac{Q(n-2)}{2} + \frac{Q(n-2)}{2}$$

That is,

$$P(n) = Q(n-1) + Q(n-2)$$

This is a $2 \times n$ grid with one particular corner (left-bottom) square missing.

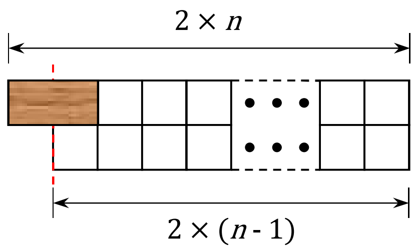


To tile it there are $\frac{Q(n)}{2}$ ways.

Let us build the recursion for $Q(n)$.

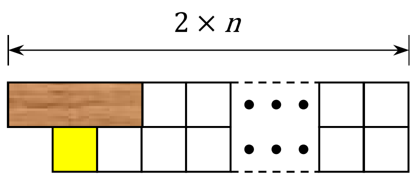
There are two cases to tile the top-left square.

Case 1:

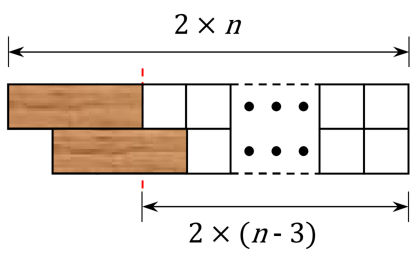


There are $\frac{Q(n-1)}{2}$ ways to tile the rest.

Case 2:



In this case there is only one way to tile the yellow square:



There are $\frac{Q(n-3)}{2}$ ways to tile the rest.

Therefore,

$$\frac{Q(n)}{2} = \frac{Q(n-1)}{2} + \frac{Q(n-3)}{2}$$

That is,

$$Q(n) = Q(n-1) + Q(n-3)$$

Now we solve the recursions:

$$P(n) = Q(n-1) + Q(n-2) \qquad (1)$$

$$Q(n) = Q(n-1) + Q(n-3) \qquad (2)$$

Since $P(n)$ is the linear sum of Qs, it has the same recursion as $Q(n)$.

So we have

$$P(n) = P(n-1) + P(n-3)$$

To see this, we plug (2) into (1):

$$P(n) = Q(n-2) + Q(n-4) + Q(n-3) + Q(n-5)$$

Note that $Q(n-2) + Q(n-3) = P(n-1)$ and $Q(n-4) + Q(n-5) = P(n-3)$.

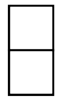
We obtain

$$P(n) = P(n-1) + P(n-3)$$

To run the recursion we need initial values: $P(1), P(2), \cdots$

Let us find them.

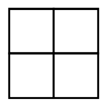
Tile 2×1 grid:



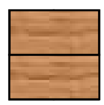
There is only one way:

$P(1) = 1$.

Tile 2×2 grid:

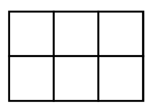


There is only one way:

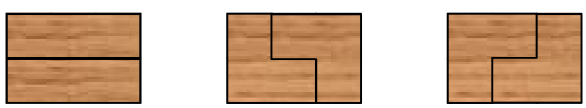


$P(2) = 1$.

Tile 2×3 grid:

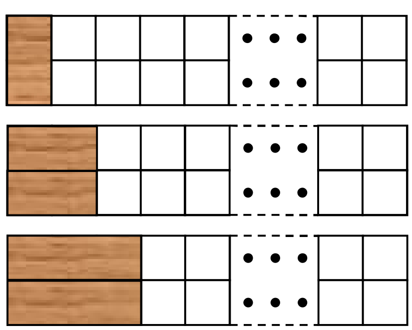


There are 3 ways:



$P(3) = 3$.

However, we cannot use them as initial values because we never tile the leftmost two squares in any of the following three ways if $n \geq 4$.

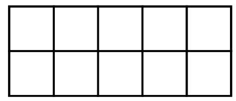


So $P(4)$ is the first valid initial value.

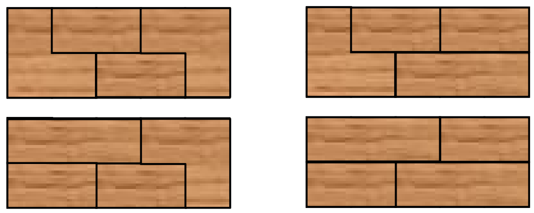
We already know $P(4) = 6$ from the problem.

Let us find $P(5)$ and $P(6)$.

Tile 2×5 grid:



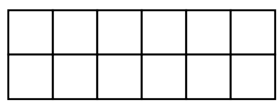
There are 8 ways:



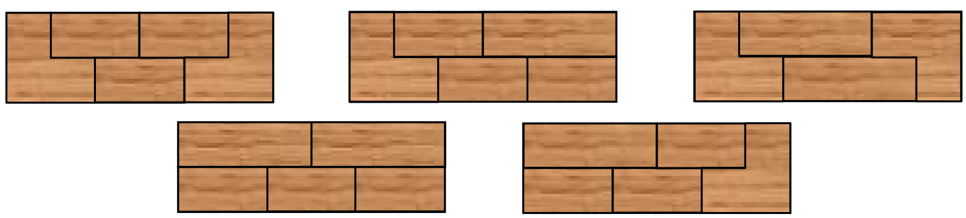
and four ways by horizontally flipping each of the above 4 tilings.

$P(5) = 8$.

Tile 2×6 grid:



There are 10 ways:



and five ways by horizontally flipping each of the above 5 tilings.

$P(6) = 10$.

The recursion

$$P(n) = P(n-1) + P(n-3)$$

is valid from $n \geq 7$.

Now we calculate $P(11)$ and $P(13)$:

$$P(7) = P(6) + P(4) = 10 + 6 = 16$$

$$P(8) = P(7) + P(5) = 16 + 8 = 24$$

$$P(9) = P(8) + P(6) = 24 + 10 = 34$$

$$P(10) = P(9) + P(7) = 34 + 16 = 50$$

$$P(11) = P(10) + P(8) = 50 + 24 = 74$$

$$P(12) = P(11) + P(9) = 74 + 34 = 108$$

$$P(13) = P(12) + P(10) = 108 + 50 = 158$$

11. (a) Find an integer $n > 1$ for which $1 + 2 + \cdots + n^2$ is a perfect square.

(b) Show that there are infinitely many integers $n > 1$ that have the property that $1 + 2 + \cdots + n^2$ is a perfect square, and determine at least three more examples of such n.

Hint: There is one approach that uses the result of a previous problem on this contest.

Answer:

(a) $7: 1 + 2 + \cdots + 7^2 = 1225 = 35^2$

(b) 41, 239, 1393

Solution:

Let $1 + 2 + \cdots + n^2 = p^2$.

We have $\frac{n^2(n^2+1)}{2} = p^2$.

If n is even, there are an odd number of 2s in the prime factorization of $\frac{n^2(n^2+1)}{2}$ and hence it cannot be a square.

So n is odd.

$\frac{n^2+1}{2}$ must be a square.

Let it be m^2: $\frac{n^2+1}{2} = m^2$.

We obtain a Pell's equation: $n^2 - 2m^2 = -1$.

To find all solutions to a Pell's equation, we need to find a solution first by trial and error.

We see that $(n, m) = (1, 1)$ is a solution.

Let $n_1 = 1$ and $m_1 = 1$.

Define n_k and m_k such that

$$n_k + \sqrt{2}m_k = (n_1 + \sqrt{2}m_1)^k \text{ for } k \geq 1.$$

Then

$$(n, m) = (n_k, m_k)\ (k \geq 1)$$

are all solutions to the Pell's equation.

We can find the recursion for (n_k, m_k).

$$n_k + \sqrt{2}m_k = (n_1 + \sqrt{2}m_1)^k = (n_1 + \sqrt{2}m_1)^{k-1}(n_1 + \sqrt{2}m_1)$$

$$= (n_{k-1} + \sqrt{2}m_{k-1})(3 + 2\sqrt{2}) = (3n_{k-1} + 4m_{k-1}) + \sqrt{2}(2n_{k-1} + 3m_{k-1})$$

We obtain

$$n_k = 3n_{k-1} + 4m_{k-1}$$

$$m_k = 2n_{k-1} + 3m_{k-1}$$

It is easy to prove that (n_k, m_k) satisfies the Pell's equation if (n_{k-1}, m_{k-1}) satisfies it. We practiced this in problem 3 of this contest.

Therefore, there are infinitively many solutions.

Calculate n_2, n_3, n_4, and n_5.

$$n_2 = 3 \cdot 1 + 4 \cdot 1 = 7, m_2 = 2 \cdot 1 + 3 \cdot 1 = 5$$

$$n_3 = 3 \cdot 7 + 4 \cdot 5 = 41, m_3 = 2 \cdot 7 + 3 \cdot 5 = 29$$

$$n_4 = 3 \cdot 41 + 4 \cdot 29 = 239, m_4 = 2 \cdot 41 + 3 \cdot 29 = 169$$

$$n_5 = 3 \cdot 239 + 4 \cdot 169 = 1393, m_5 = 2 \cdot 239 + 3 \cdot 169 = 985$$

The first four solutions for $n > 1$ are:

$$1 + 2 + \cdots + 7^2 = \frac{7^2(7^2 + 1)}{2} = 7^2 \cdot 5^2 = 35^2$$

$$1 + 2 + \cdots + 41^2 = \frac{41^2(41^2 + 1)}{2} = 41^2 \cdot 29^2 = 1189^2$$

$$1 + 2 + \cdots + 239^2 = \frac{239^2(239^2 + 1)}{2} = 239^2 \cdot 169^2 = 40391^2$$

$$1 + 2 + \cdots + 1393^2 = \frac{1393^2(1393^2 + 1)}{2} = 1393^2 \cdot 985^2 = 1372105^2$$